

General Bezout-type theorems

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Abstract

In this sequel to [9] we develop Bezout type theorems for semidegrees (including an explicit formula for *iterated semidegrees*) and an inequality for subdegrees. In addition we prove (in case of surfaces) a Bernstein type theorem for the number of solutions of two polynomials in terms of the mixed volume of planar convex polygons associated to them (via the theory of Kaveh-Khovanskii [5] and Lazarsfeld-Mustata [6])

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1 Introduction

Disclaimer: This is an unpolished draft of the article. A clearer exposition (with more complete reference) is in order and this submission will be updated in a few days.

This article is a sequel to [9]. In it we develop affine Bezout type theorems. In Section 3 we show that given a polynomial system of n equations on an n -dimensional affine variety, there are subdegrees which add nothing at infinity to generic fibers. In Section ?? we find estimates for the number of solutions of the system in terms of the properties of the subdegree. The estimate is exact if the subdegree turns out to be a semidegree. If in addition the semidegree is of a special class called *iterated* semidegrees, then the formula turns out to be explicit and this is handled in Section 4. Finally, in Section 5, we show that the estimate for subdegrees is exact if $n = 2$. We also give an interpretation of this estimate in terms of the mixed volume of planar convex polygons associated to the subdegrees (via the theory of Kaveh-Khovanskii [5] and Lazarsfeld-Mustata [6]).

2 Existence of Intersection Preserving Filtrations

Let X be an affine variety over \mathbb{K} . Recall that for subsets V_1, \dots, V_m of X , a completion $\psi : X \hookrightarrow Z$ is said to *preserve the intersection of V_1, \dots, V_m at ∞* if $\overline{V}_1 \cap \dots \cap \overline{V}_m \cap X_\infty = \emptyset$, where $X_\infty := Z \setminus X$ is the set of ‘points at infinity’ and \overline{V}_j is the closure of V_j in Z for every j .

Lemma 2.1. *Let $\mathcal{F} = \{F_d : d \geq 0\}$ be a complete filtration on $A := \mathbb{K}[X]$, and $\psi_{\mathcal{F}} : X \hookrightarrow X^{\mathcal{F}} := \text{Proj } A^{\mathcal{F}}$ be the corresponding completion.*

1. *For each ideal \mathfrak{q} of A , let $\mathfrak{q}^{\mathcal{F}} := \bigoplus_{d \geq 0} (\mathfrak{q} \cap F_d) \subseteq A^{\mathcal{F}}$. Then the closure of $V(\mathfrak{q}) \subseteq X$ in $X^{\mathcal{F}}$ is $V(\mathfrak{q}^{\mathcal{F}})$.*
2. *Let V_1, \dots, V_m be Zariski closed subsets of X with $V_i = V(\mathfrak{q}_i)$ for ideals $\mathfrak{q}_i \subseteq A$ for each i . Let \mathcal{I} be the ideal of $A^{\mathcal{F}}$ generated by $\mathfrak{q}_1^{\mathcal{F}}, \dots, \mathfrak{q}_m^{\mathcal{F}}$ and $(1)_1$. Then $\psi_{\mathcal{F}}$ preserves the intersection of V_1, \dots, V_m at ∞ iff the $\sqrt{\mathcal{I}} \supseteq A_+^{\mathcal{F}}$, where $A_+^{\mathcal{F}} := \bigoplus_{d > 0} F_d$ is the irrelevant ideal of $A^{\mathcal{F}}$.*

Proof. 1. Recall (example ??) that there exists $d > 0$ such that $(A^{\mathcal{F}})^{[d]} := \bigoplus_{k \geq 0} F_{kd}$ is generated by F_d as a \mathbb{K} -algebra. Define a new filtration $\mathcal{G} := \{G_k : k \geq 0\}$ on A by $G_k := F_{kd}$. Let $\{1, g_1, \dots, g_m\}$ be a \mathbb{K} -vector space basis of G_1 . Then $A^{\mathcal{G}} \cong (A^{\mathcal{F}})^{[d]}$ and by corollary ??, $X^{\mathcal{G}} := \text{Proj } A^{\mathcal{G}}$ is the closure in $\mathbb{P}^m(\mathbb{K})$ of $\phi(X)$, where $\phi : X \rightarrow \mathbb{K}^m$ is defined by: $\phi(x) := (g_1(x), \dots, g_m(x))$.

Let \mathfrak{q} be an ideal of A and $V := V(\mathfrak{q})$ be the Zariski closed subset of X defined by \mathfrak{q} . Let $\mathfrak{p} := \ker \phi^*$ and $\mathfrak{r} := (\phi^*)^{-1}(\mathfrak{q})$, where $\phi^* : \mathbb{K}[y_1, \dots, y_m] \rightarrow A$ is the pull back by means of ϕ . Identify X with $V(\mathfrak{p})$ and V with $V(\mathfrak{r})$ in \mathbb{K}^m . Then $X^{\mathcal{G}}$ and the closure $\overline{V}^{\mathcal{G}}$ of V in $X^{\mathcal{G}}$ are the Zariski closed subsets of $\mathbb{P}^m(\mathbb{K})$ determined by the *homogenizations* $\tilde{\mathfrak{p}}$ of \mathfrak{p} and, respectively, $\tilde{\mathfrak{r}}$ of \mathfrak{r} with respect to y_0 .

Moreover, the closed embedding $\Phi : X^{\mathcal{G}} \hookrightarrow \mathbb{P}^m(\mathbb{K})$ is induced by the surjective homomorphism $\Phi^* : \mathbb{K}[y_0, \dots, y_m] \rightarrow A^{\mathcal{G}}$ which maps $y_0 \mapsto (1)_1$ and $y_i \mapsto (g_i)_1$ for $1 \leq i \leq m$. Therefore $\overline{V}^{\mathcal{G}}$ in $X^{\mathcal{G}}$ is defined by the ideal $\Phi^*(\tilde{\mathfrak{r}})$. But the d -th graded component of $\Phi^*(\tilde{\mathfrak{r}})$ is

$$\begin{aligned} \Phi^*((\tilde{\mathfrak{r}})_d) &:= \{\Phi^*(\tilde{f}(y_0, \dots, y_m)) : \tilde{f} \in \tilde{\mathfrak{r}}, \tilde{f} \text{ homogeneous}, \deg(\tilde{f}) = d\} \\ &= \{\tilde{f}((1)_1, (g_1)_1, \dots, (g_m)_1) : \\ &\quad \tilde{f} \text{ homogeneous in } y_0, \dots, y_m, \deg(\tilde{f}) = d, \tilde{f}(1, y_1, \dots, y_m) \in \mathfrak{r}\} \\ &= \{(\tilde{f}(1, g_1, \dots, g_m))_d : \\ &\quad \tilde{f} \text{ homogeneous in } y_0, \dots, y_m, \deg(\tilde{f}) = d, \tilde{f}(1, g_1, \dots, g_m) \in \mathfrak{q}\} \\ &= \{(f(g_1, \dots, g_m))_d : \\ &\quad f \text{ polynomial in } y_1, \dots, y_m, \deg(f) \leq d, f(g_1, \dots, g_m) \in \mathfrak{q}\} \\ &= \{(g)_d : g \in \mathfrak{q} \cap G_d\}, \end{aligned}$$

where the last equality is a consequence of $A^{\mathcal{G}} = \mathbb{K}[(1)_1, (g_1)_1, \dots, (g_m)_1]$. Then $\Phi^*(\tilde{\mathfrak{r}}) = \bigoplus_{d \geq 0} \Phi^*((\tilde{\mathfrak{r}})_d) = \bigoplus_{d \geq 0} \{(g)_d : g \in \mathfrak{q} \cap G_d\} = \mathfrak{q}^{\mathcal{G}}$. Now recall (example ??) that $X^{\mathcal{F}}$ and $X^{\mathcal{G}}$ are isomorphic and this isomorphism is induced by the inclusion $A^{\mathcal{G}} \subseteq A^{\mathcal{F}}$. Since $\mathfrak{q}^{\mathcal{F}} \cap A^{\mathcal{G}} = \mathfrak{q}^{\mathcal{G}}$, it follows that the Zariski closed subset of $X^{\mathcal{G}}$ determined by $\mathfrak{q}^{\mathcal{G}}$ is isomorphic to the Zariski closed subset of $X^{\mathcal{F}}$ determined by $\mathfrak{q}^{\mathcal{F}}$.

2. $\psi_{\mathcal{F}}$ preserves the intersection of V_1, \dots, V_m at ∞ iff $\overline{V}_1 \cap \dots \cap \overline{V}_m \cap X_\infty = \emptyset$. By part 1 $\overline{V}_j = V(\mathfrak{q}_j^{\mathcal{F}})$ for each j , and by theorem ?? $X_\infty = V((1)_1)$. Therefore $\overline{V}_1 \cap \dots \cap \overline{V}_m \cap X_\infty$ is

determined by the ideal of $A^\mathcal{F}$ generated by $(1)_1, \mathfrak{q}_1^\mathcal{F}, \dots, \mathfrak{q}_m^\mathcal{F}$, which is precisely the definition of \mathcal{I} . Then the projective version of Nullstellensatz (see section ??) implies $V(\mathcal{I}) = \emptyset$ iff $A_+^\mathcal{F} \subseteq \sqrt{\mathcal{I}}$. \square

Theorem 2.2 (see [7, Theorem 1.2(1)] and [8, Theorem 1.3.1]). *Let V_1, \dots, V_m be Zariski closed subsets in an affine variety X such that $\cap_{i=1}^m V_i$ is a finite set. Then there is a complete filtration \mathcal{F} on $\mathbb{K}[X]$ such that $\psi_\mathcal{F}$ preserves the intersection of the V_i 's at ∞ .*

Proof. Let $X \subseteq \mathbb{K}^n$ and the ideals in $\mathbb{K}[x_1, \dots, x_n]$ defining X, V_1, \dots, V_m be respectively $\mathfrak{p}, \mathfrak{q}_1, \dots, \mathfrak{q}_m$ with $\mathfrak{q}_j \supseteq \mathfrak{p}$ for each j .

Claim. *For each $i = 1, \dots, n$, there is an integer $d_i \geq 1$ such that*

$$x_i^{d_i} = f_{i,1} + \dots + f_{i,m} + g_i \quad (1)$$

for some $f_{i,j} \in \mathfrak{q}_j$ and a polynomial $g_i \in \mathbb{K}[x_i]$ of degree less than d_i .

Proof. If $V_1 \cap \dots \cap V_m = \emptyset$, then by Nullstellensatz $\langle \mathfrak{q}_1, \dots, \mathfrak{q}_m \rangle$ is the unit ideal in $\mathbb{K}[x_1, \dots, x_n]$, and the claim is trivially satisfied with $g_i := 0$ for each i . So assume

$$V_1 \cap \dots \cap V_m = \{P_1, \dots, P_k\} \subseteq \mathbb{K}^n,$$

for some $k \geq 1$. Let $P_i = (a_{i,1}, \dots, a_{i,n}) \in \mathbb{K}^n$. For each $i = 1, \dots, n$, let

$$h_i := (x_i - a_{1,i})(x_i - a_{2,i}) \cdots (x_i - a_{k,i}).$$

By Nullstellensatz, for some $d'_i \geq 1$, $h_i^{d'_i} \in \langle \mathfrak{q}_1, \dots, \mathfrak{q}_m \rangle$, i.e. $h_i^{d'_i} = f_{i,1} + \dots + f_{i,m}$ for some $f_{i,j} \in \mathfrak{q}_j$. Substituting $h_i = \prod_j (x_i - a_{j,i})$ in the preceding equation we see that the claim holds with $d_i := kd'_i$. \square

Below for $S \subseteq \mathbb{K}[X]$ we denote by $\mathbb{K}\langle S \rangle$ the \mathbb{K} -linear span of S , and for an element $g \in \mathbb{K}[x_1, \dots, x_n]$, we denote by \bar{g} the image of g in $\mathbb{K}[X] = \mathbb{K}[x_1, \dots, x_n]/\mathfrak{p}$. Fix a set of $f_{i,j}$'s satisfying the conclusion of the previous claim. Then define a filtration \mathcal{F} on $\mathbb{K}[X]$ as follows: let

$$\begin{aligned} F_0 &:= \mathbb{K}, \\ F_1 &:= \mathbb{K}\langle 1, \bar{x}_1, \dots, \bar{x}_n, \bar{f}_{1,1}, \dots, \bar{f}_{n,m} \rangle, \\ F_k &:= F_1^k \text{ for } k > 1, \\ \mathcal{F} &:= \{F_i : i \geq 0\}. \end{aligned}$$

Clearly \mathcal{F} is a complete filtration. We now show that this \mathcal{F} satisfies the conclusion of the theorem. By lemma 2.1 this is equivalent to showing that $\sqrt{\mathcal{I}} \supseteq \mathbb{K}[X]_+^\mathcal{F}$, where \mathcal{I} is the ideal generated by $\bar{\mathfrak{q}}_1^\mathcal{F}, \dots, \bar{\mathfrak{q}}_m^\mathcal{F}$ and $(1)_1$ in $\mathbb{K}[X]^\mathcal{F}$.

From the construction of \mathcal{F} it follows that $\mathbb{K}[X]_+^\mathcal{F}$ is generated by the elements $(1)_1, (\bar{x}_1)_1, \dots, (\bar{x}_n)_1, (\bar{f}_{1,1})_1, \dots, (\bar{f}_{n,m})_1$. Note that $\bar{f}_{i,j} \in \bar{\mathfrak{q}}_j$ for each i, j , so that $(\bar{f}_{i,j})_1 \in \bar{\mathfrak{q}}_j^\mathcal{F} \subseteq \mathcal{I}$. Moreover, $(1)_1 \in \mathcal{I}$. So, all we really need to show is that $(\bar{x}_i)_1 \in \sqrt{\mathcal{I}}$ for all $i = 1, \dots, n$.

Reducing equation (1) mod \mathfrak{p} , we have $(\bar{x}_i)^{d_i} = \bar{f}_{i,1} + \dots + \bar{f}_{i,m} + \bar{g}_i \in \mathbb{K}[X]$ for all $i = 1, \dots, n$. Let $g_i = \sum_{j=0}^{d_i-1} a_{i,j} x_i^j$. Then in $\mathbb{K}[X]^{\mathcal{F}}$,

$$((\bar{x}_i)_1)^{d_i} = ((1)_1)^{d_i-1}((\bar{f}_{i,1})_1 + \dots + (\bar{f}_{i,m})_1) + \sum_{j=0}^{d_i-1} a_{i,j}((\bar{x}_i)_1)^j((1)_1)^{d_i-j}.$$

All of the summands in the right hand side lie inside \mathcal{I} , hence $((\bar{x}_i)_1)^{d_i} \in \mathcal{I}$ for all $i = 1, \dots, n$, as required. \square

Recall that given a polynomial map $f = (f_1, \dots, f_q) : X \rightarrow \mathbb{K}^q$, $a = (a_1, \dots, a_q) \in \mathbb{K}^q$ and a completion ψ of X , ψ *preserves* $\{f_1, \dots, f_n\}$ at ∞ over a if ψ preserves the intersection of the hypersurfaces $H_i(a) := \{x \in X : f_i(x) = a_i\}$, $i = 1, \dots, q$.

Example 2.3. Consider map $f : \mathbb{K}^2 \rightarrow \mathbb{K}^2$ given by $f(x, y) := (x, y + x^3)$. For $a := (a_1, a_2) \in \mathbb{K}^2$,

$$\begin{aligned} H_1(a) &= \{(a_1, y) : y \in \mathbb{K}\}, \\ H_2(a) &= \{(x, a_2 - x^3) : x \in \mathbb{K}\}. \end{aligned}$$

We claim that in the usual completion $\mathbb{P}^2(\mathbb{K})$ of \mathbb{K}^2 , the closures of $H_1(a)$ and $H_2(a)$ intersect at a point P at infinity for each $a \in \mathbb{K}^2$, and hence $\mathbb{P}^2(\mathbb{K})$, as the natural completion of \mathbb{K}^2 , does not preserve $\{f_1, \dots, f_n\}$ at ∞ over any point of \mathbb{K}^2 .

Indeed, write the homogeneous coordinates of $\mathbb{P}^2(\mathbb{K})$ as $[z : x : y]$ and identify \mathbb{K}^2 with $\mathbb{P}^2(\mathbb{K}) \setminus V(z)$. Let $a \in \mathbb{K}^2$. When $\mathbb{K} = \mathbb{C}$, the ‘infinite’ points in $\overline{H_i}(a)$ can be described as the limits of points in $H_i(a)$. Therefore, the points at infinity of $H_1(a)$ are $\lim_{|y| \rightarrow \infty} [1 : a_1 : y] = \lim_{|y| \rightarrow \infty} [1/y : a_1/y : 1] = [0 : 0 : 1]$. Similarly, the infinite part of $H_2(a)$ is $\lim_{|x| \rightarrow \infty} [1 : x : a_2 - x^3] = \lim_{|x| \rightarrow \infty} [1/(a_2 - x^3) : x/(a_2 - x^3) : 1] = [0 : 0 : 1]$, and hence the claim is true with $P := [0 : 0 : 1]$.

To verify the claim for an arbitrary \mathbb{K} , one has to apply lemma 2.1 with $X = \mathbb{K}^2$ and calculate $\overline{H_i}(a) \cap X_\infty = V(\mathfrak{q}_i^{\mathcal{F}}(a), (1)_1)$, where $\mathfrak{q}_i(a)$ is the ideal of $H_i(a)$. A straightforward calculation shows: the graded ring $\mathbb{K}[X]^{\mathcal{F}}$ corresponding to the embedding $\mathbb{K}^2 \hookrightarrow \mathbb{P}^2(\mathbb{K})$ is isomorphic to $\mathbb{K}[x, y, z]$ where z plays the role of $(1)_1$, and $\mathfrak{q}_i^{\mathcal{F}}(a)$ is the homogenization of $\mathfrak{q}_i(a)$ with respect to z . Then $\mathfrak{q}_1(a) = \langle x - a_1 \rangle$ and $\mathfrak{q}_2(a) = \langle y + x^3 - a_2 \rangle$, so that $\mathfrak{q}_1^{\mathcal{F}}(a) = \langle x - a_1 z \rangle$ and $\mathfrak{q}_2^{\mathcal{F}}(a) = \langle y z^2 + x^3 - a_2 z^3 \rangle$. Therefore $\overline{H_1}(a) \cap X_\infty = V(x - a_1 z, z) = V(x, z) = \{[0 : 0 : 1]\}$. Similarly $\overline{H_2}(a) \cap X_\infty = V(y z^2 + x^3 - a_2 z, z) = V(x, z) = \{[0 : 0 : 1]\}$ and the claim is valid with the same P as in $\mathbb{K} = \mathbb{C}$ case.

Because it is simpler to describe, we will from now on frequently use only the limit argument (valid only for $\mathbb{K} = \mathbb{C}$) in order to find the points at infinity of various subvarieties of a given X . In all these cases, the analogous results also follow over an arbitrary algebraically closed field \mathbb{K} by means of straightforward calculations (and if $\text{char } \mathbb{K} = 0$, by Tarski-Lefschetz principle).

We now find, following the proof of Theorem 2.2, a completion of \mathbb{K}^2 which preserves $\{f_1, \dots, f_n\}$ at ∞ over 0. In the notation of theorem 2.2, $\mathfrak{q}_1 = \langle x \rangle$ and $\mathfrak{q}_2 = \langle y + x^3 \rangle$. Observe that $x \in \mathfrak{q}_1$, and y satisfies

$$y = -x^3 + (y + x^3),$$

with $x^3 \in \mathfrak{q}_1$ and $y + x^3 \in \mathfrak{q}_2$. Let filtration $\mathcal{F} := \{F_i : i \geq 0\}$ on $\mathbb{K}[x, y]$ be defined as follows: $F_0 := \mathbb{K}$, $F_1 := \mathbb{K}\langle 1, x, y, x^3 \rangle$, and $F_k := (F_1)^k$ for $k > 1$. Then as in the proof of theorem 2.2,

completion $\psi_{\mathcal{F}}$ preserves $\{f_1, \dots, f_n\}$ at ∞ over 0. Let us now show this directly. By corollary ??, the corresponding completion $X^{\mathcal{F}}$ is isomorphic to the closure in $\mathbb{P}^3(\mathbb{K})$ of the image of $\phi : \mathbb{K}^2 \rightarrow \mathbb{P}^3(\mathbb{K})$, where $\phi(x, y) := [1 : x : y : x^3]$. Then $\phi(H_1(a)) = \{[1 : a_1 : y : a_1^3] : y \in \mathbb{K}\}$ and limit $\lim_{y \rightarrow \infty} [1 : a_1 : y : a_1^3] = \lim_{y \rightarrow \infty} [1/y : a_1/y : 1 : a_1^3/y] = [0 : 0 : 1 : 0]$, for $a \in \mathbb{K}^2$, so that the only point at infinity of $\overline{H_1(a)}$ is $[0 : 0 : 1 : 0]$. Similarly, $\phi(H_2(a)) = \{[1 : x : a_2 - x^3 : x^3] : x \in \mathbb{K}\}$ and $\lim_{x \rightarrow \infty} [1 : x : a_2 - x^3 : x^3] = \lim_{x \rightarrow \infty} [1/x^3 : 1/x^2 : (a_2 - x^3)/x^3 : 1] = [0 : 0 : -1 : 1]$. Therefore $\overline{H_2(a)}$ also has only one point at infinity and it is $[0 : 0 : -1 : 1]$. It follows that $\overline{H_1(a)} \cap \overline{H_2(a)} \cap X_{\infty} = \emptyset$ for all a , i.e. $X^{\mathcal{F}}$ preserves $\{f_1, \dots, f_n\}$ at ∞ over *every* point of \mathbb{K}^2 .

Example 2.4. Let $f(x, y) := (x, y)$ on \mathbb{K}^2 . Then for each $a = (a_1, a_2) \in \mathbb{K}^2$, $H_1(a) = \{(a_1, y) : y \in \mathbb{K}\}$ and $H_2(a) = \{(x, a_2) : x \in \mathbb{K}\}$. Consider filtration \mathcal{F} on $\mathbb{K}[x, y]$ defined by: $F_0 := \mathbb{K}$, $F_1 := \mathbb{K}\langle 1, x, y, xy, x^2y^2 \rangle$, and $F_k := (F_1)^k$ for $k \geq 2$. By corollary ??, $X^{\mathcal{F}}$ is the closure of the image of \mathbb{K}^2 under the map $\phi : \mathbb{K}^2 \hookrightarrow \mathbb{P}^4(\mathbb{K})$ defined by: $\phi(x, y) = [1 : x : y : xy : x^2y^2]$. Then $\phi(H_1(a)) = \{[1 : a_1 : y : a_1y : a_1^2y^2] : y \in \mathbb{K}\}$. If $a_1 = 0$, then $\phi(H_1(a)) = \{[1 : 0 : y : 0 : 0] : y \in \mathbb{K}\}$, and hence the only point at infinity in $\overline{\phi(H_1(a))}$ is $[0 : 0 : 1 : 0 : 0]$. But if $a_1 \neq 0$, then dividing all coordinates by $a_1^2y^2$, we see that the point at infinity in $\overline{\phi(H_1(a))}$ is $[0 : 0 : 0 : 0 : 1]$. Similarly, $\phi(H_2(a)) = \{[1 : x : a_2 : y : a_2x : a_2^2x^2] : x \in \mathbb{K}\}$ and the only point at infinity in $\overline{\phi(H_2(a))}$ is $[0 : 1 : 0 : 0 : 0]$ if $a_2 = 0$, and $[0 : 0 : 0 : 0 : 1]$ if $a_2 \neq 0$. Therefore $X^{\mathcal{F}}$ preserves $\{f_1, \dots, f_n\}$ at ∞ over a iff $a_1 = 0$ or $a_2 = 0$, i.e. iff a belongs to the union of the coordinate axes.

Let $f : X \rightarrow Y$ be a generically finite map of affine varieties of the same dimension. Given any $y \in Y$ such that $f^{-1}(y)$ is finite, theorem 2.2 guarantees the existence of a projective completion of X that preserves $\{f_1, \dots, f_n\}$ at ∞ over y . But as the preceding example shows, the completion might fail to preserve $\{f_1, \dots, f_n\}$ at ∞ over ‘most of the’ points in the image of f . This suggests that we should look for a completion ψ which preserves $\{f_1, \dots, f_n\}$ at ∞ over y for *generic* $y \in Y$, i.e. ψ preserves $\{f_1, \dots, f_n\}$ at ∞ . We will demonstrate two ways to accomplish this goal - we start with a simpler-to-prove theorem 2.5 and will present a stronger version in theorem 2.7 following (cf. [7, Theorem 1.2] and [8, Theorem 1.3.4]).

Theorem 2.5. * *Let $f : X \rightarrow Y \subseteq \mathbb{K}^q$ be a generically finite map of affine varieties of same dimension. Include \mathbb{K}^q into $(\mathbb{P}^1(\mathbb{K}))^q$ via the componentwise inclusion $(a_1, \dots, a_q) \mapsto ([1 : a_1], \dots, [1 : a_q])$. Let $\phi : X \hookrightarrow Z$ be any completion of X . Define \bar{X} to be the closure of the graph of f in $Z \times (\mathbb{P}^1(\mathbb{K}))^q$. Then \bar{X} preserves $\{f_1, \dots, f_n\}$ at ∞ . If ϕ comes from some filtration on $\mathbb{K}[X]$, then there is a filtration \mathcal{F} on $\mathbb{K}[X]$ and a commuting diagram as follows:*

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \bar{X} & \xrightarrow{\cong} & X^{\mathcal{F}} \end{array}$$

Proof. Let $\pi := (\pi_1, \dots, \pi_q) : Z \times (\mathbb{P}^1(\mathbb{K}))^q \rightarrow (\mathbb{P}^1(\mathbb{K}))^q$ be the natural projection. Then π maps \bar{X} onto the closure \bar{Y} of Y in $(\mathbb{P}^1(\mathbb{K}))^q$. Let $V := Z \setminus X$. Then $\tilde{V} := (V \times (\mathbb{P}^1(\mathbb{K}))^q) \cap \bar{X}$ is a proper Zariski closed subset of \bar{X} . Since Z is complete, it follows that $\pi(\tilde{V})$ is a proper Zariski closed subset of \bar{Y} . We now show that for all $y \in Y \setminus \pi(\tilde{V})$, \bar{X} preserves $\{f_1, \dots, f_n\}$ at ∞ over y .

*The idea of looking at the construction of theorem 2.5 is due to Professor A. Khovanskii.

Pick an arbitrary $y := (y_1, \dots, y_q) \in Y$ such that \bar{X} does not preserve $\{f_1, \dots, f_n\}$ at ∞ over y . It suffices to show that $y \in \pi(\tilde{V})$. As usual, let $H_i(y) := \{x \in X : f_i(x) = y_i\}$. By assumption there is $\tilde{x} \in \bar{H}_1(y) \cap \dots \cap \bar{H}_q(y) \cap (\bar{X} \setminus X)$, where for each k , $\bar{H}_k(y)$ is the closure of $H_k(y)$ in \bar{X} . Fix a k , $1 \leq k \leq q$. Note that $\pi_k(H_k(y)) = \{y_k\}$. By continuity of π_k it follows that $\pi_k(\bar{H}_k(y)) = \{y_k\}$. But then $\pi_k(\tilde{x}) = y_k$. It follows that $\pi(\tilde{x}) = y$ and hence \tilde{x} is of the form (z, y) for some $z \in Z$. We claim that z does not lie in X . Indeed, if $z \in X$, it would imply $\tilde{x} \in (X \times Y) \cap (\bar{X} \setminus \psi(X))$, where $\psi : X \hookrightarrow \bar{X}$ is the inclusion. Consider the chain of inclusions: $\psi(X) \subseteq X \times Y \subseteq Z \times (\mathbb{P}^1(\mathbb{K}))^q$. Note:

1. $\psi(X)$ is the graph of f in $X \times Y$, and hence is a Zariski closed subvariety of $X \times Y$.
2. $X \times Y$ is Zariski open in $Z \times (\mathbb{P}^1(\mathbb{K}))^q$.
3. If $T \subseteq U \subseteq W$ are topological spaces such that T is closed in U and U is open in W , then $\bar{T} \cap U = T$, where \bar{T} is the closure of T in W .

Since \bar{X} is by definition the closure of $\psi(X)$ in $Z \times (\mathbb{P}^1(\mathbb{K}))^q$, it follows via the above observations that $\bar{X} \cap (X \times Y) = \psi(X)$, so that $(X \times Y) \cap (\bar{X} \setminus \psi(X)) = \emptyset$. This contradiction proves the claim. It follows that $z \in Z \setminus X = V$. Then $\tilde{x} \in \tilde{V}$. Therefore $y = \pi(\tilde{x}) \in \pi(\tilde{V})$ and the first claim of the theorem is proved.

As for the last claim, note that if ϕ comes from a filtration, then by corollary ?? we may assume that $Z \subseteq \mathbb{P}^p(\mathbb{K})$ for some p and $\phi(x) = [1 : g_1(x) : \dots : g_p(x)]$ for some $g_1, \dots, g_p \in \mathbb{K}[X]$. Hence the inclusion $\psi : X \hookrightarrow \bar{X}$ is of the form: $\psi(x) = ([1 : g_1(x) : \dots : g_p(x)], [1 : f_1(x)], \dots, [1 : f_q(x)])$. Let $l := (p+1)2^q - 1$ and let us embed $\mathbb{P}^p(\mathbb{K}) \times (\mathbb{P}^1(\mathbb{K}))^q \hookrightarrow \mathbb{P}^l(\mathbb{K})$ via the *Segre* embedding s which maps $w := ([w_0 : \dots : w_p], [w_{1,0} : w_{1,1}], \dots, [w_{q,0} : w_{q,1}])$ to the point $s(w)$ whose homogeneous coordinates are monomials of degree $q+1$ in w of the form $w_i w_{1,j_1} w_{2,j_2} \dots w_{q,j_q}$ where $0 \leq i \leq p$ and $0 \leq j_k \leq 1$ for each k . The component of $s \circ \psi$ corresponding to $i = j_1 = \dots = j_q = 0$ is 1 and hence $s \circ \psi$ maps $x \in X$ to a point with homogeneous coordinates $[1 : h_1(x) : \dots : h_l(x)]$ for some $h_1, \dots, h_l \in \mathbb{K}[X]$. Then corollary ?? implies that there is a filtration \mathcal{F} on $\mathbb{K}[X]$ such that the closure \bar{X} of $s \circ \psi(X)$ in $\mathbb{P}^l(\mathbb{K})$ is isomorphic to $X^{\mathcal{F}}$ via an isomorphism which is identity on X . Morphism s being an isomorphism completes the proof. \square

Remark 2.6. Let $f : X \rightarrow Y$ be a map of n -dimensional affine varieties with generically finite fibers and ψ be a completion of X . Define $S_\psi := \{a \in f(X) : \psi \text{ preserves } \{f_1, \dots, f_n\} \text{ at } \infty \text{ over } y\}$. It will be interesting to know if S_ψ has any intrinsic structure. In example 2.4 S_ψ was the union of two coordinate axes in \mathbb{K}^2 , and hence a proper closed subset of $f(X)$. On the other hand, theorem 2.5 shows that there are completions ψ of X such that S_ψ contains a dense open subset of $f(X)$. We now give an example where S_ψ is indeed a proper dense open subset of $f(X)$, namely:

Let $X = Y = \mathbb{C}^2$ and $f : X \rightarrow Y$ be the map defined by $f_1 := x_1^3 + x_1^2 x_2 + x_1 x_2^2 - x_2$ and $f_2 := x_1^3 + 2x_1^2 x_2 + x_1 x_2^2 - x_2$. It is easy to see that f is quasifinite. Let $\phi : X \hookrightarrow \mathbb{P}^2(\mathbb{C})$ be the usual completion, and let $\psi : X \hookrightarrow \bar{X}$ be as in theorem 2.4. Let the coordinates of $\mathbb{P}^2(\mathbb{C})$ be $[Z : X_1 : X_2]$. Identify X with $\mathbb{P}^2(\mathbb{C}) \setminus V(Z)$, so that $x_i = X_i/Z$ for $i = 1, 2$. Then $X \ni (x_1, x_2) \xrightarrow{\psi} ([1 : x_1 : x_2], [1 : f_1(x)], [1 : f_2(x)])$. We claim that $f(X) \setminus S_\psi$ is the line $L := \{(c, c) : c \in \mathbb{C}\}$.

Indeed, let $a := (a_1, a_2) \in Y$. Define, as usual, $H_i(a) := \{x \in X : f_i(x) = a_i\}$ for $i = 1, 2$. Let $C_i(a)$ be the closure in $\mathbb{P}^2(\mathbb{C})$ of $H_i(a)$ for each i . It is easy to see that $P := [0 : 0 : 1] \in C_1(a) \cap C_2(a)$. Choose local coordinates $\xi_1 := X_1/X_2$ and $\xi_2 := Z/X_2$ of

$\mathbb{P}^2(\mathbb{C})$ near $P := [0 : 0 : 1]$. Equations of $f_{i,a}$ in (ξ_1, ξ_2) coordinates are:

$$\begin{aligned} f_{1,a} &= \xi_1^3 + \xi_1^2 + \xi_1 - \xi_2^2 - a_1 \xi_2^3 \\ f_{2,a} &= \xi_1^3 + 2\xi_1^2 + \xi_1 - \xi_2^2 - a_2 \xi_2^3 \end{aligned} \quad (2)$$

It follows that for each i , $C_i(a)$ is smooth at P (in particular, each has only one branch at P) and both admit parametrizations at P of the form

$$\gamma_{i,a}(t) := [t : t^2 + o(t^3) : 1], \quad (3)$$

where $o(t^3)$ means terms of order t^3 and higher. In (x_1, x_2) coordinates the parametrizations are of the form: $(x_1(t), x_2(t)) = (t + o(t^2), 1/t)$ for $t \neq 0$. Since $f_2(x) = f_1(x) + x_1^2 x_2$, it follows that for $t \neq 0$,

$$\begin{aligned} \psi(\gamma_{1,a}(t)) &= (\gamma_{1,a}(t), [1 : a_1], [1 : a_1 + \frac{(t + o(t^2))^2}{t}]) \\ &= (\gamma_{1,a}(t), [1 : a_1], [1 : a_1 + t + o(t^2)]) \end{aligned}$$

Therefore $\lim_{t \rightarrow 0} \psi(\gamma_{1,a}(t)) = (P, [1 : a_1], [1 : a_1])$. Since $f_1(x) = f_2(x) - x_1^2 x_2$, the same argument also gives $\lim_{t \rightarrow 0} \psi(\gamma_{2,a}(t)) = (P, [1 : a_2], [1 : a_2])$.

To summarize, we proved that if $a \in L$, then $(P, [1 : a_1], [1 : a_1]) \in \bar{H}_1(a) \cap \bar{H}_2(a) \cap X_\infty$, where as usual $\bar{H}_i(a)$ is the closure of $H_i(a)$ in \bar{X} and $X_\infty := \bar{X} \setminus X$. It follows that $L \subseteq f(X) \setminus S_\psi$.

To prove the other inclusion, assume $a \in f(X) \setminus S_\psi$. Pick $z \in \bar{H}_1(a) \cap \bar{H}_2(a) \cap X_\infty$. Then $z = (Q, [1 : a_1], [1 : a_2])$ for a point $Q \in \mathbb{P}^2(\mathbb{C})$. Therefore it follows that $Q \in (C_1(a) \setminus H_1(a)) \cap (C_2(a) \setminus H_2(a))$, where curves $C_i(a)$ are the closures in $\mathbb{P}^2(\mathbb{C})$ of $H_i(a)$, $i = 1, 2$. But the only possible choice for such point Q is point P . Therefore limit $\lim_{t \rightarrow 0} \psi(\gamma_{1,a}(t)) = z = \lim_{t \rightarrow 0} \psi(\gamma_{2,a}(t))$, which implies that $(P, [1 : a_1], [1 : a_1]) = (P, [1 : a_2], [1 : a_2])$. Therefore $a_1 = a_2$ and $a \in L$, as claimed.

Let $f : X \rightarrow Y \subseteq \mathbb{K}^q$ be as in theorem 2.5. In the theorem following we find completions with even stronger preservation property at ∞ , namely completions that *preserve map f at ∞* (remark-definition ??).

Theorem 2.7 (cf. [7, Theorem 1.2(2)] and [8, Theorem 1.3.4]). *Let $f : X \rightarrow Y \subseteq \mathbb{K}^q$ be a generically finite map of affine varieties of the same dimension. Then there is a complete filtration \mathcal{F} on the coordinate ring of X such that $\psi_{\mathcal{F}}$ preserves map f at ∞ .*

Proof. Choose a set of coordinates x_1, \dots, x_p (resp. y_1, \dots, y_q) of X (resp. Y). Since f is generically finite, it follows that the coordinate ring of X is algebraic over the pullback of the coordinate ring of Y . In particular, each x_i satisfies a polynomial of the form:

$$\sum_{j=0}^{k_i} g_{i,j}(y)(x_i)^j = 0 \quad (4)$$

for some $k_i \geq 1$ and regular functions $g_{i,j}(y)$ on Y such that $g_{i,k_i} \neq 0 \in \mathbb{K}[Y]$. In abuse of notation, but for the sake of convenience, we implicitly identified in (4) variables y_k with polynomials f_k for each k . We continue to do so throughout this proof. Let $g_{i,j}(y) = \sum_{\alpha} c_{i,j,\alpha} y^\alpha$ be an arbitrary representation of $g_{i,j}$ in $\mathbb{K}[Y]$. For each i, j with $1 \leq i \leq p$ and

$0 \leq j \leq k_i$, let $d_{i,j} := \deg_y(\sum_{\alpha} c_{i,j,\alpha} y^{\alpha}) := \max\{|\alpha| : c_{i,j,\alpha} \neq 0\}$, where $\alpha = (\alpha_1, \dots, \alpha_q) \in (\mathbb{Z}_+)^q$ and $|\alpha| := \alpha_1 + \dots + \alpha_q$. Let $d_0 := \max\{d_{i,0} : 1 \leq i \leq p\}$ and $k_0 := \max\{k_i : 1 \leq i \leq p\}$. Define a filtration $\mathcal{F} := \{F_i : i \geq 0\}$ on $\mathbb{K}[X]$ as follows:

$$\begin{aligned} F_0 &:= \mathbb{K}, \\ F_1 &:= \mathbb{K}\langle 1, x_1, \dots, x_p, y_1, \dots, y_q \rangle + \mathbb{K}\langle y^{\beta} : |\beta| \leq d_0 \rangle + \\ &\quad + \mathbb{K}\langle x_i y^{\beta} : |\beta| \leq d_{i,1}, 1 \leq i \leq p \rangle, \\ F_k &:= \begin{cases} \sum_{j=1}^{k-1} F_j F_{k-j} + \mathbb{K}\langle (x_i)^k y^{\beta} : |\beta| \leq d_{i,k}, 1 \leq i \leq p \rangle & \text{if } 1 < k \leq k_0, \\ \sum_{j=1}^{k-1} F_j F_{k-j} & \text{if } k > k_i \forall i. \end{cases} \end{aligned}$$

Let $g := \prod_{i=1}^m g_{i,k_i}$ and $U := \{a \in Y : g(a) \neq 0\}$. Then U is a non-empty Zariski open subset of Y . Let $\xi := (\xi_1, \dots, \xi_q) : \mathbb{K}^q \rightarrow \mathbb{K}^q$ be an arbitrary linear change of coordinates of \mathbb{K}^q . It suffices to show that $\psi_{\mathcal{F}}$ preserves the components of $\xi \circ f$ at ∞ over $\xi(a)$ for $a := (a_1, \dots, a_q) \in U$. For each $a \in Y$, let $H_j(a) := \{x \in X : (\xi_j \circ f)(x) = \xi_j(a)\}$ and let $\mathfrak{q}_j(a)$ be the ideal of $H_j(a)$, i.e. the ideal of $\mathbb{K}[X]$ generated by $\xi_j(y) - \xi_j(a)$. By lemma 2.1, $\psi_{\mathcal{F}}$ preserves the components of $\xi \circ f$ at infinity over $\xi(a)$ iff $\sqrt{\mathcal{I}(a)} = \mathbb{K}[X]_+^{\mathcal{F}}$, where $\mathcal{I}(a)$ is the ideal of $\mathbb{K}[X]^{\mathcal{F}}$ generated by $\mathfrak{q}_1^{\mathcal{F}}(a), \dots, \mathfrak{q}_n^{\mathcal{F}}(a)$ and $(1)_1$. Note the following:

- (a) Since ξ is a linear change of coordinate, so is ξ^{-1} . Therefore, for all $d \geq 0$, the \mathbb{K} -span of $\{y^{\beta} : |\beta| \leq d\}$ in $\mathbb{K}[Y]$ is equal to the \mathbb{K} -span of $\{(\xi^{-1}(y))^{\beta} : \deg_y((\xi^{-1}(y))^{\beta}) \leq d\}$.
- (b) If we replace f by $\xi \circ f$, and hence y by $\xi(y)$, then $g_{i,j}(y)$ changes to $g_{i,j}^{\xi}(y) := g_{i,j}(\xi^{-1}(y)) = \sum_{\alpha} c_{i,j,\alpha} (\xi^{-1}(y))^{\alpha}$. But replacing $\sum_{\alpha} c_{i,j,\alpha} y^{\alpha}$ by $\sum_{\alpha} c_{i,j,\alpha} (\xi^{-1}(y))^{\alpha}$ does *not* change its degree $d_{i,j}$ in y .
- (c) Let $g^{\xi} := \prod_{i=1}^m g_{i,k_i}^{\xi}$. Then $g^{\xi}(\xi(a)) \neq 0$ iff $g(a) \neq 0$.

In view of the latter observations and the construction of \mathcal{F} it follows that \mathcal{F} does not change if we replace f by $\xi \circ f$. Moreover, the following two claims are equivalent due to properties (a), (b) and (c) of the preceding paragraph.

- (1) $\psi_{\mathcal{F}}$ preserves the components of f at ∞ over $a \in U$, and
- (2) $\psi_{\mathcal{F}}$ preserves the components of $\xi \circ f$ at ∞ over $a \in \xi^{-1}(U)$.

Therefore it suffices to prove (1) and we may without loss of generality assume ξ to be the identity. Note that $\mathbb{K}[X]_+^{\mathcal{F}}$ is generated as a \mathbb{K} -algebra by elements $(1)_1, (x_1)_1, \dots, (x_p)_1, (y_1)_1, \dots, (y_q)_1$, the $(y^{\beta})_1$'s that appear in the definition of F_1 , and all those $((x_i)^k y^{\beta})_k$ that we inserted in the definition of all F_k 's. Therefore $\sqrt{\mathcal{I}(a)} = \mathbb{K}[X]_+^{\mathcal{F}}$ iff some power of each of these generators lies in $\mathcal{I}(a)$.

Lemma 2.7.1. *Let a be an arbitrary point in Y .*

- 1. *Let $\beta \in (\mathbb{Z}_+)^q$ be such that $y^{\beta} \in F_1$. then*
 - (a) *$(y-a)^{\beta}$ also lies in F_1 , and*
 - (b) *$((y-a)^{\beta})_1 \in \mathcal{I}(a)$.*
- 2. *Let $1 \leq i \leq p$. Pick k with $1 \leq k \leq k_i$ and $\beta \in (\mathbb{Z}_+)^q$ such that $x_i^k y^{\beta} \in F_k$. Then*
 - (a) *$x_i^k (y-a)^{\beta}$ lies in F_k , and*
 - (b) *if in addition $\beta \neq 0$, then $(x_i^k (y-a)^{\beta})_k \in \mathcal{I}(a)$.*

Proof. 1. Pick $\beta \in (\mathbb{Z}_+)^q$ such that $y^{\beta} \in F_1$. Expanding $(y-a)^{\beta}$ in powers of y_1, \dots, y_q , we see that $(y-a)^{\beta} = \sum_{|\gamma| \leq |\beta|} c_{\gamma} y^{\gamma}$ for some $c_{\gamma} \in \mathbb{K}$ and $\gamma \in (\mathbb{Z}_+)^q$. By construction, F_1 contains each of the y^{γ} appearing in the preceding expression. It follows that F_1 also contains $(y-a)^{\beta}$, which proves assertion 1a. As for 1b, note that if $\beta = 0$, then $((y-a)^{\beta})_1 = (1)_1 \in \mathcal{I}(a)$.

Otherwise, there exists j , $1 \leq j \leq q$, such that the j -th coordinate of β is positive. Then $(y-a)^\beta \in \mathfrak{q}_j(a)$, and hence $((y-a)^\beta)_1 \in \mathfrak{q}_j^{\mathcal{F}}(a) \subseteq \mathcal{I}(a)$, which completes the proof of assertion 1.

2. Let $1 \leq i \leq p$. Pick k, β such that $1 \leq k \leq k_i$ and $x_i^k y^\beta \in F_k$. As in the proof of assertion 1, expanding $(y-a)^\beta$ in powers of y_1, \dots, y_q , we see that $x_i^k (y-a)^\beta = \sum_{|\gamma| \leq |\beta|} c_\gamma x_i^k y^\gamma$ for some $c_\gamma \in \mathbb{K}$. By construction, F_k contains $x_i^k y^\gamma$ for each $|\gamma| \leq |\beta|$. It follows that F_k also contains $x_i^k (y-a)^\beta$, and this proves assertion 2a. For 2b, note that if $\beta \neq 0$, then there exists j , $1 \leq j \leq q$, such that the j -th coordinate of β is positive. Then $x_i^k (y-a)^\beta \in \mathfrak{q}_j(a)$, and hence $(x_i^k (y-a)^\beta)_k \in \mathfrak{q}_j^{\mathcal{F}}(a) \subseteq \mathcal{I}(a)$, which completes the proof of the lemma. \square

We now return to the proof of theorem 2.7. Let a be any point in Y and β be such that $y^\beta \in F_1$. Expanding y^β in powers of $y_1 - a_1, \dots, y_q - a_q$, we see that $y^\beta = \sum_{|\gamma| \leq |\beta|} c'_\gamma (y-a)^\gamma$ for some $c'_\gamma \in \mathbb{K}$. By assertion 1a of lemma 2.7.1, each $(y-a)^\gamma$ in the preceding expression lies in F_1 . Therefore, in $\mathbb{K}[X]^{\mathcal{F}}$ element $(y^\beta)_1 = \sum_{|\gamma| \leq |\beta|} c'_\gamma ((y-a)^\gamma)_1$. But assertion 1b of lemma 2.7.1 implies that $((y-a)^\gamma)_1 \in \mathcal{I}(a)$ for all $|\gamma| \leq |\beta|$. Therefore $(y^\beta)_1 \in \mathcal{I}(a)$.

Now expand the polynomial of the left hand side of equation (4) (as a polynomial in $y := (y_1, \dots, y_q)$) in powers of $y_1 - a_1, \dots, y_q - a_q$. This leads to an equation of the form

$$\sum_{j=0}^{k_i} g_{i,j}(a) x_i^j + \sum_{\substack{j \leq k_i \\ \beta \neq 0}} h_{i,j,\beta}(a) (y-a)^\beta x_i^j = 0, \quad (4')$$

where terms $(y-a)^\beta x_i^j$, for $\beta \neq 0$ and $j \leq k_i$, that appear in (4') are such that $y^\beta x_i^j \in F_j$, and due to assertion 2a of lemma 2.7.1, $(y-a)^\beta x_i^j$ are in $F_j \subseteq F_{k_i}$. Therefore equation (4') implies the following equality in $\mathbb{K}[X]^{\mathcal{F}}$:

$$g_{i,k_i}(a) ((x_i)_1)^{k_i} = - \sum_{j=0}^{k_i-1} g_{i,j}(a) ((x_i)_1)^j ((1)_1)^{k_i-j} - \sum_{\substack{j \leq k_i \\ \beta \neq 0}} h_{i,j,\beta}(a) ((y-a)^\beta x_i^j)_{k_i}. \quad (4'')$$

Since $(1)_1 \in \mathcal{I}(a)$, each of the terms under the first summation of the right hand side of (4'') lies in $\mathcal{I}(a)$. Moreover, by assertion 2b of lemma 2.7.1, every $((y-a)^\beta x_i^j)_{k_i}$ under the second summation of the right hand side of (4'') also belongs to $\mathcal{I}(a)$. It follows that $g_{i,k_i}(a) ((x_i)_1)^{k_i} \in \mathcal{I}(a)$, and therefore $(x_i)_1 \in \sqrt{\mathcal{I}(a)}$ if $g_{i,k_i}(a) \neq 0$. Hence for all $a \in U$ and i , $1 \leq i \leq p$, elements $(x_i)_1 \in \sqrt{\mathcal{I}(a)}$.

Let $a \in U$ and $1 \leq i \leq p$. Pick $(x_i)^k y^\beta \in F_k$ with $1 \leq k \leq k_i$. Expanding y^β in powers of $y_1 - a_1, \dots, y_q - a_q$, we see that $(x_i)^k y^\beta = \sum_{|\gamma| \leq |\beta|} c'_\gamma (x_i)^k (y-a)^\gamma$ for some $c'_\gamma \in \mathbb{K}$. By assertion 2a of lemma 2.7.1, each $(x_i)^k (y-a)^\gamma$ in the preceding expression lies in F_k . Therefore in $\mathbb{K}[X]^{\mathcal{F}}$ element $((x_i)^k y^\beta)_k = \sum_{|\gamma| \leq |\beta|} c'_\gamma ((x_i)^k (y-a)^\gamma)_k$. If $|\gamma| \leq |\beta|$ and $\gamma \neq 0$, assertion 2b of lemma 2.7.1 implies that $((x_i)^k (y-a)^\gamma)_k \in \mathcal{I}(a)$ and if $\gamma = 0$, then $((x_i)^k (y-a)^\gamma)_k = ((x_i)^k)_k = ((x_i)_1)^k \in \sqrt{\mathcal{I}(a)}$ due to the conclusion of the preceding paragraph. Therefore for $x_i^k y^\beta \in F_k$, $1 \leq k \leq k_i$, it follows that $((x_i)^k y^\beta)_k \in \sqrt{\mathcal{I}(a)}$.

Consequently $\sqrt{\mathcal{I}(a)} = \mathbb{K}[X]_+^{\mathcal{F}}$ for all $a \in U$, which completes the proof. \square

Example 2.8. Let $X = Y = \mathbb{C}^2$ and $f := (f_1, f_2) : X \rightarrow Y$ be the map defined by $f_1 := x_1^3 + x_1^2 x_2 + x_1 x_2^2 - c x_2$ and $f_2 := x_1^3 + x_1^2 x_2 + x_1 x_2^2 - x_2$ for any complex number $c \neq 0$ or 1. Note that this map is a minor variation of the map considered in remark 2.6.

Let $\psi : X \hookrightarrow \bar{X} \subseteq \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the completion considered in theorem 2.5 and remark 2.6, i.e. $\psi(x_1, x_2) := ([1 : x_1 : x_2], [1 : f_1(x_1, x_2)], [1 : f_2(x_1, x_2)])$ for all $(x_1, x_2) \in X$. Below we show that ψ does *not* satisfy the preservation property of theorem 2.7.

For each $\lambda := (\lambda_1, \lambda_2) \in \mathbb{C}^2$ let

$$f_\lambda := \lambda_1 f_1 + \lambda_2 f_2 = (\lambda_1 + \lambda_2)x_1^3 + (\lambda_1 + \lambda_2)x_1^2 x_2 + (\lambda_1 + \lambda_2)x_1 x_2^2 - (\lambda_1 c + \lambda_2)x_2.$$

Fix $\lambda^1, \lambda^2 \in \mathbb{C}^2 \setminus (L_1 \cup L_2 \cup L_3)$ where L_1 is the x_1 -axis, L_2 is the x_2 -axis and L_3 is the line $\{(x_1, x_2) \in \mathbb{C}^2 : x_1 + x_2 = 0\}$. Fix an $a := (a_1, a_2) \in Y$. Let $b_i := \lambda_1^i a_1 + \lambda_2^i a_2$, $i = 1, 2$. Every $H_i(a) = \{x \in X : f_{\lambda^i}(x) = b_i\}$ is as in theorem 2.7, $i = 1, 2$. Let $C_i(a)$ be the closure in $\mathbb{P}^2(\mathbb{C})$ of $H_i(a)$ for $i = 1, 2$. Then $P := [0 : 0 : 1] \in C_1(a) \cap C_2(a)$, where as in remark 2.6, we choose coordinates $[Z : X_1 : X_2]$ on $\mathbb{P}^2(\mathbb{C})$ and identify X with $\mathbb{P}^2(\mathbb{C}) \setminus V(Z)$. Choose local coordinates $\xi_1 := X_1/X_2$ and $\xi_2 := Z/X_2$ on $\mathbb{P}^2(\mathbb{C})$ near $P := [0 : 0 : 1]$. Equations of $f_{\lambda^i}(x_1, x_2) - b_i$ in (ξ_1, ξ_2) coordinates are:

$$f_{\lambda^i}(x_1, x_2) - b_i = (\lambda_1^i + \lambda_2^i)\xi_1^3 + (\lambda_1^i + \lambda_2^i)\xi_1^2 + (\lambda_1^i + \lambda_2^i)\xi_1 - (\lambda_1^i c + \lambda_2^i)\xi_2^2 - a_i \xi_2^3.$$

Since $\lambda_1^i + \lambda_2^i \neq 0$, it follows (similarly to the implication (2) \Rightarrow (3) of remark 2.6) that for each i , $C_i(a)$ is smooth at P and has a parametrization at P of the form

$$\gamma_{i,a}(t) := [t : \frac{\lambda_1^i c + \lambda_2^i}{\lambda_1^i + \lambda_2^i} t^2 + o(t^3) : 1].$$

In (x_1, x_2) coordinates the parametrizations are: $(x_1(t), x_2(t)) = (\frac{\lambda_1^i c + \lambda_2^i}{\lambda_1^i + \lambda_2^i} t + o(t^2), 1/t)$ for $t \in \mathbb{C}^*$. A straightforward calculation shows that for $t \in \mathbb{C}^*$

$$\begin{aligned} f_1(\gamma_{i,a}(t)) &= \frac{\lambda_2^i(1-c)}{\lambda_1^i + \lambda_2^i} \frac{1}{t} + o(t^2) \quad \text{and} \\ f_2(\gamma_{i,a}(t)) &= \frac{\lambda_1^i(c-1)}{\lambda_1^i + \lambda_2^i} \frac{1}{t} + o(t^2), \end{aligned} \tag{5}$$

$i = 1, 2$. By our choice of $c \neq 0, 1$ and λ^i 's off $L_1 \cup L_2 \cup L_3$ it follows that the coefficients at $\frac{1}{t}$ in the expressions in (5) are non-zero. It follows that $\lim_{t \rightarrow 0} |f_j(\gamma_{i,a}(t))| = \infty$ for each i, j , and therefore

$$\begin{aligned} \lim_{t \rightarrow 0} \psi(\gamma_{i,a}(t)) &= \lim_{t \rightarrow 0} ([1 : \gamma_{i,a}(t)], [1 : f_1(\gamma_{i,a}(t))], [1 : f_2(\gamma_{i,a}(t))]) \\ &= ([0 : 0 : 1], [0 : 1], [0 : 1]). \end{aligned}$$

Hence $([0 : 0 : 1], [0 : 1], [0 : 1])$ is in the closure of $H_i(a)$ in \bar{X} for $i = 1, 2$ and every $a \in Y$. It follows that ψ does not preserve $(f_{\lambda^1}, f_{\lambda^2})$ at ∞ over *any* point in Y , as claimed.

Finally, we follow the proof of theorem 2.7 to find a completion which preserves map f at ∞ . The algebraic equations satisfied by x_1 and x_2 over $\mathbb{C}[f_1, f_2]$ are:

$$\begin{aligned} x_1^3 + \frac{1}{1-c}(f_1 - f_2)x_1^2 + \frac{1}{(1-c)^2}(f_1 - f_2)^2 x_1 - \frac{1}{1-c}(f_1 - c f_2) &= 0, \text{ and} \\ x_2 - \frac{1}{1-c}(f_1 - f_2) &= 0. \end{aligned}$$

In the notations of the proof of theorem 2.7, $d_{1,2} = d_{1,0} = 1$, $d_{1,1} = 2$ and $d_{2,0} = 1$. Let $\mathcal{F} := \{F_d : d \geq 0\}$ be the filtration defined by:

$$F_0 := \mathbb{K}, F_1 := \mathbb{K}\langle 1, x_1, x_2, f_1, f_2, x_1 f_1, x_1 f_2, x_1 f_1^2, x_1 f_1 f_2, x_1 f_2^2 \rangle, F_d := (F_1)^d \text{ for } d \geq 2.$$

Construction of the proof of 2.7 yields that $X^{\mathcal{F}}$ preserves map f at ∞ . Indeed, $\psi_{\mathcal{F}}(x) = [1 : x_1 : x_2 : f_1(x) : f_2(x) : x_1 f_1(x) : x_1 f_2(x) : x_1 f_1^2(x) : x_1 f_1(x) f_2(x) : x_1 f_2^2(x)] \in \mathbb{P}^9(\mathbb{C})$ for $x \in \mathbb{C}^2$ and therefore applying (5) it follows that $\lim_{t \rightarrow 0} \psi_{\mathcal{F}}(\gamma_{i,a}(t)) =$

$$\begin{aligned} & \lim_{t \rightarrow 0} [1 : \left(\frac{\lambda_1^i c + \lambda_2^i}{\lambda_1^i + \lambda_2^i} t + o(t^2) \right) : \frac{1}{t} : \left(\frac{\lambda_2^i (1-c)}{\lambda_1^i + \lambda_2^i} \frac{1}{t} + o(t^2) \right) : \left(\frac{\lambda_1^i (c-1)}{\lambda_1^i + \lambda_2^i} \frac{1}{t} + o(t^2) \right) : o(1) : \\ & \quad o(1) : \left(\frac{(1-c)^2 (\lambda_2^i)^2 (\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3} \frac{1}{t} + o(t^2) \right) : \left(-\frac{(1-c)^2 \lambda_1^i \lambda_2^i (\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3} \frac{1}{t} + o(t^2) \right) : \\ & \quad \left(\frac{(1-c)^2 (\lambda_1^i)^2 (\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3} \frac{1}{t} + o(t^2) \right)] \\ & = [0 : 0 : 1 : \frac{\lambda_2^i (1-c)}{\lambda_1^i + \lambda_2^i} : \frac{\lambda_1^i (c-1)}{\lambda_1^i + \lambda_2^i} : 0 : 0 : \frac{(1-c)^2 (\lambda_2^i)^2 (\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3} : \\ & \quad - \frac{(1-c)^2 \lambda_1^i \lambda_2^i (\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3} : \frac{(1-c)^2 (\lambda_1^i)^2 (\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3}] . \end{aligned}$$

If λ^1 and λ^2 are linearly independent, then it follows that limits $\lim_{t \rightarrow 0} \psi_{\mathcal{F}}(\gamma_{i,a}(t))$ are different for $i = 1, 2$, and therefore, completion $\psi_{\mathcal{F}}$ preserves $\{f_{\lambda^1}, f_{\lambda^2}\}$ at ∞ over every $a \in Y = \mathbb{C}^2$.

3 General Bezout-type theorems

3.1 Bezout theorem for Semidegrees

Let X be an n dimensional affine variety. Let δ be a complete degree like function on the coordinate ring A of X with associated filtration $\mathcal{F} := \{F_d : d \geq 0\}$. Recall from example ?? that there exists $d > 0$ such that $(A^\delta)^{[d]} := \bigoplus_{k \geq 0} F_{kd}$ is generated by F_d as a \mathbb{K} -algebra and then the d -uple embedding embeds X^δ into $\mathbb{P}^l(\mathbb{K})$, where $l := \dim_{\mathbb{K}} F_d - 1$.

Below we will make use of a notion of multiplicity at an isolated point b of fiber $f^{-1}(a)$ of morphism $f : X \rightarrow \mathbb{K}^n$, where X is an affine variety and $a := (a_1, \dots, a_n) \in \mathbb{K}^n$. The latter multiplicity we define following the definition in [2, Example 12.4.8] as the intersection multiplicity at b of the effective Cartier divisors determined by regular (on X) functions $f_j - a_j$, $1 \leq j \leq n$.

Theorem 3.1 (see [7, Theroem 1.3] and [8, Theorem 3.1.1]). *Let X , A , δ , d and l be as above. Denote by D the degree of X^δ in $\mathbb{P}^l(\mathbb{K})$. Let $f = (f_1, \dots, f_n) : X \rightarrow \mathbb{K}^n$ be any generically finite map. Then for all $a \in \mathbb{K}^n$,*

$$|f^{-1}(a)| \leq \frac{D}{d^n} \prod_{i=1}^n \delta(f_i) \quad (\text{A})$$

where $|f^{-1}(a)|$ is the number of the isolated points in fiber $f^{-1}(a)$ each counted with the multiplicity of $f^{-1}(a)$ at the respective point. If in addition δ is a semidegree and ψ_δ preserves $\{f_1, \dots, f_n\}$ at ∞ over a , then (A) holds with an equality.

Proof. Let $a = (a_1, \dots, a_n) \in \mathbb{K}^n$. For each i , let \mathfrak{a}_i be the ideal generated by $(f_i - a_i)^d$ in A and $\mathfrak{a}_i^\delta := \bigoplus_{j \geq 0} (\langle (f_i - a_i)^d \rangle \cap F_j)$ be the corresponding *homogeneous* ideal in A^δ . Clearly $G_i := ((f_i - a_i)^d)_{dd_i} \in \mathfrak{a}_i$, where $d_i := \delta(f_i)$ for each $i = 1, \dots, n$. It follows that

$$\bigcap_{i=1}^n \{x \in X : (f_i(x) - a_i)^d = 0\} \subseteq \bigcap_{i=1}^n \{x \in X^\delta : G(x) = 0 \text{ for all } G \in \mathfrak{a}_i^\delta\} \quad (6)$$

$$\subseteq \bigcap_{i=1}^n \{x \in X^\delta : G_i(x) = 0\} \quad (7)$$

where the first inclusion is due to lemma 2.1, since the closure of the hypersurface $V(\mathfrak{a}_i)$ of X in X^δ is $V(\mathfrak{a}_i^\delta)$. Note that the sum of the multiplicities of the intersections of Cartier divisors determined by $(f_i - a_i)^d$ at the isolated points in the set on the left hand side of (6) is precisely d^n times the sum of the multiplicities of fiber $f^{-1}(a)$ at the isolated points in $f^{-1}(a)$.

Pick a set of homogeneous coordinates $[y_0 : \dots : y_L]$ of $\mathbb{P}^l(\mathbb{K})$. The d -uple embedding of X^δ into $\mathbb{P}^l(\mathbb{K})$ induces a surjective homomorphism $\phi : \mathbb{K}[y_0, \dots, y_L] \rightarrow (A^\delta)^{[d]}$ of graded \mathbb{K} -algebras. For each i , choose an arbitrary homogeneous polynomial $\hat{G}_i \in \mathbb{K}[y_0, \dots, y_L]$ of degree d_i such that $\phi(\hat{G}_i) = G_i$. According to the classical Bezout theorem on $\mathbb{P}^l(\mathbb{K})$, the sum of the multiplicities of isolated points of $X^\delta \cap V(\hat{G}_1) \cap \dots \cap V(\hat{G}_n)$ in $\mathbb{P}^l(\mathbb{K})$ is at most $Dd_1 \cdots d_n$, and it is equal to $Dd_1 \cdots d_n$ if all the points in the intersection are isolated.

Since $X^\delta \cap V(\hat{G}_1) \cap \dots \cap V(\hat{G}_n)$ is precisely $X^\delta \cap V(G_1) \cap \dots \cap V(G_n)$, inequality (A) follows from (7) and the conclusions of the two preceding paragraphs. The last assertion of theorem 3.1 follows from the following observations:

1. if δ is a semidegree, then according to lemma ?? ideal \mathfrak{a}_i^δ is generated by G_i and hence \subseteq in (7) can be replaced by $=$, and
2. completion ψ_δ preserves $\{f_1, \dots, f_n\}$ at ∞ over a iff the \subseteq in (6) is in fact an equality. \square

Remark 3.2. Both assertions of theorem 3.1 are valid for $\delta = \max_{j=1}^N \delta_j$ being a subdegree with $\delta_1(f_i) = \dots = \delta_N(f_i)$ for all $i = 1, \dots, n$.

Remark 3.3. Let $f := (f_1, \dots, f_n) : X \rightarrow \mathbb{K}^n$ be a quasifinite map and δ be a complete semidegree on $A := \mathbb{K}[X]$. We claim that if ψ_δ preserves $\{f_1, \dots, f_n\}$ at ∞ over any point, then ψ_δ preserves $\{f_1, \dots, f_n\}$ at ∞ over *all* points. Indeed, let $a := (a_1, \dots, a_n) \in \mathbb{K}^n$. For each i , let $\mathfrak{p}_i(a)$ be the ideal generated by $(f_i - a_i)$ in A , and let $\mathfrak{p}_i^\delta(a) := \bigoplus_{j \geq 0} (\mathfrak{p}_i(a) \cap F_j) \subseteq A^\delta$. Due to lemma 2.1, ψ_δ preserves $\{f_1, \dots, f_n\}$ at ∞ over a iff $\sqrt{\mathcal{I}(a)} = A_+^\delta$, where $\mathcal{I}(a) := \langle \mathfrak{p}_1^\delta(a), \dots, \mathfrak{p}_n^\delta(a), (1)_1 \rangle \subseteq A^\delta$. According to lemma ??, $\mathfrak{p}_i^\delta(a) = \langle (f_i - a_i)_{d_i} \rangle$, where $d_i := \delta(f_i)$ for each i . Then $\mathcal{I}(a) = \langle (f_1 - a_1)_{d_1}, \dots, (f_n - a_n)_{d_n}, (1)_1 \rangle = \langle (f_1)_{d_1}, \dots, (f_n)_{d_n}, (1)_1 \rangle$. Since the latter expression is independent of a_i 's, the claim follows. Note that we have shown (in the notation of remark 2.6) that either $S_{\psi_\delta} = \emptyset$ or $S_{\psi_\delta} = f(X)$.

Example 3.4 (Weighted Bezout theorem, cf. [1], [7, Example 7], [8, Example 3.1.4]). Let $X := \mathbb{K}^n$ and δ be a weighted degree on $A := \mathbb{K}[x_1, \dots, x_n]$ which assigns weights $d_i > 0$ to x_i , $1 \leq i \leq n$. Then $(1)_1, (x_1)_{d_1}, \dots, (x_n)_{d_n}$ is a set of \mathbb{K} -algebra generators of A^δ . A straightforward application of lemma 2.1 implies that ψ_δ preserves at ∞ all components of the identity map $\mathbf{1}$ of \mathbb{K}^n over 0. Therefore theorem 3.1 with $f = \mathbf{1}$ and d as in the preamble to theorem 3.1 implies that $1 = \frac{D}{d^n} \prod_{i=1}^n d_i$ and therefore $D = \frac{d^n}{\prod_{i=1}^n d_i}$. Consequently, theorem

3.1 implies for any f that:

$$|f^{-1}(a)| \leq \prod_{i=1}^n \frac{\delta(f_i)}{d_i}. \quad (8)$$

Recall (example ??) that $\text{gr } A^\delta \cong \mathbb{K}[x_1, \dots, x_n]$ via an identification of $[(g)_{\delta(g)}] \in \text{gr } A^\delta$ and $\mathfrak{L}_\delta(g) \in \mathbb{K}[x_1, \dots, x_n]$, for $0 \neq g \in A$. Let $a := (a_1, \dots, a_n) \in \mathbb{K}^n$ and $\mathfrak{a}_i := \langle f_i - a_i \rangle$ (cf. the proof of theorem 3.1). According to lemma ??, ideals \mathfrak{a}_i^δ are generated by $(f_i - a_i)_{\delta(f_i)} = (f_i)_{\delta(f_i)} - a_i((1)_1)^{\delta(f_i)}$ for $1 \leq i \leq n$. Therefore,

$$\begin{aligned} (8) \text{ holds with equality} &\iff \psi_\delta \text{ preserves } \{f_1, \dots, f_n\} \text{ at } \infty \text{ over } a \\ &\iff V(\mathfrak{a}_1^\delta, \dots, \mathfrak{a}_n^\delta, (1)_1) = \emptyset \subseteq \text{Proj } A^\delta \\ &\iff V((f_1)_{\delta(f_1)}, \dots, (f_n)_{\delta(f_n)}, (1)_1) = \emptyset \subseteq \text{Proj } A^\delta \\ &\iff V([(f_1)_{\delta(f_1)}], \dots, [(f_n)_{\delta(f_n)}]) = \emptyset \subseteq \text{Proj gr } A^\delta \\ &\iff V(\mathfrak{L}_\delta(f_1), \dots, \mathfrak{L}_\delta(f_n)) = \{0\} \in \mathbb{K}^n, \end{aligned}$$

where the last relation holds since the maximal ideal of the origin in \mathbb{K}^n corresponds to the irrelevant ideal $\bigoplus_{k>0} F_k/F_{k-1}$ of $\text{gr } A^\delta$ via the isomorphism $\text{gr } A^\delta \cong \mathbb{K}[x_1, \dots, x_n]$. Finally note that formula (8) in combination with condition $V(\mathfrak{L}_\delta(f_1), \dots, \mathfrak{L}_\delta(f_n)) = \{0\} \in \mathbb{K}^n$ for equality in (8) constitute the content of the *Weighted Bezout theorem* stated in section ??.

Applying theorem ??, number D of theorem 3.1 admits a geometric description as the volume of a convex body associated to δ and a \mathbb{Z}^n valued (surjective) valuation ν of $\mathbb{K}(X)$ (cf. section ??), namely:

Proposition 3.5 (see [7, Proposition 1.4] and [8, Theorem 3.1.3]). *Let X , A , d , δ , l and D be as in theorem 3.1 and ν be a valuation on A with values in \mathbb{Z}^n . Let C be the smallest closed cone in \mathbb{R}^{n+1} containing*

$$G := \{(\frac{1}{d}\delta(f), \nu(f)) : f \in A\} \cup \{(1, 0, \dots, 0)\}.$$

Let Δ be the convex hull of the cross-section of C with the first coordinate having value 1. Then $D = n! \text{Vol}_n(\Delta)$, where Vol_n is the n -dimensional Euclidean volume.

Proof. Consider $L := F_d = \{f \in A : \delta(f) \leq d\}$. As in section ??, we introduce $S(L) := \{(k, \nu(f)) : f \in L^k, k \geq 0\} \subseteq \mathbb{N} \oplus \mathbb{Z}^n$. Let $C(L)$ be the closure of the convex hull of $S(L)$ in \mathbb{R}^{n+1} and $\Delta(L) := C(L) \cap (\{1\} \times \mathbb{R}^n)$. Mapping Φ_L of theorem ?? is precisely the embedding $X \subseteq X^\delta \hookrightarrow \mathbb{P}^l(\mathbb{K})$, so that the mapping degree $d(L)$ of Φ_L is 1. Therefore theorem ?? implies that $[L, \dots, L] = \frac{n!}{s(L)} \text{Vol}_n(\Delta(L))$, where $s(L)$ is the index in \mathbb{Z}^n of the subgroup $S'(L)$ generated by all the differences $\alpha - \beta$ such that $(k, \alpha), (k, \beta) \in S(L)$ for some $k \geq 1$. Note also that:

1. according to its definition $[L, \dots, L]$ is equal to the degree D of X^δ in $\mathbb{P}^L(\mathbb{C})$,
2. F_d generates $(A^\delta)^{[d]}$, so that for $L^k = (F_d)^k = F_{kd}$ for all $k \geq 1$, and therefore for all $f \in A$, $(k, \nu(f)) \in S(L)$ for all $k \geq \frac{\delta(f)}{d}$, and
3. for all $k \geq 1$, $1 \in L^k$ and therefore $(k, \nu(1)) = (k, 0, \dots, 0) \in S(L)$.

Properties 2 and 3 imply that $S'(L) \supseteq \{\nu(f) : f \in A\}$. Since ν is surjective, it follows that $S'(L) = \mathbb{Z}^n$ and hence $s(L) = 1$. But then $D = n! \text{Vol}_n(\Delta(L))$ due to property 1. Finally, properties 2 and 3 also imply that $C = C(L)$, and consequently that $\Delta = \Delta(L)$. \square

Example 3.6. Let $X = \mathbb{K}^n$ and δ be as in example 3.4. Let ν be any monomial valuation on $A := \mathbb{K}[x_1, \dots, x_n]$, e.g. the one that assigns to $\sum a_\alpha x^\alpha \in A \setminus \{0\}$ the lexicographically (coordinatewise) minimal exponent α among all $\alpha := (\alpha_1, \dots, \alpha_n)$ with $a_\alpha \neq 0$. With d being any common multiple of d_1, \dots, d_n , it is straightforward to see that $\Delta = \{(1, x) \in \mathbb{R}_+^{n+1} : \sum_{i=1}^n x_i d_i \leq d\}$ and therefore $\text{Vol}_n(\Delta) = \frac{1}{n!} \prod_{i=1}^n \frac{d}{d_i}$. It follows that $D = n! \text{Vol}_n(\Delta) = \prod_{i=1}^n \frac{d}{d_i}$, which is, of course, the value we have calculated in example 3.4.

3.2 Bezout theorem for Subdegrees

Assume $f := (f_1, \dots, f_n) : X \rightarrow \mathbb{K}^n$ is a dominating morphism of affine varieties with generically finite fibers and $\delta := \max\{\delta_j : 1 \leq j \leq N\}$ is a complete subdegree on $A := \mathbb{K}[X]$, i.e. δ is non-negative, finitely generated and $\delta^{-1}(0) = \mathbb{K} \setminus \{0\}$. Assume $\delta_j(f_i) > 0$ for each i, j . In the spirit of theorem 3.1 we derive in this section an upper bound for the number of points in a generic fiber of f in X (counted with multiplicity) in terms of the degree of a projective completion of X .

Definition.

- Let $g \in A$ and $\text{div}_X(g)$ be the principal Cartier divisor corresponding to g on X . Assume that the corresponding Weil divisor is $[\text{div}_X(g)] = \sum r_i [V_i]$. Given a completion $X \hookrightarrow Y$ of X , we write $[\overline{\text{div}}_X^Y(g)]$ for the Weil divisor on Y given by:

$$[\overline{\text{div}}_X^Y(g)] := \sum r_i [\overline{V}_i],$$

where \overline{V}_i is the closure of V_i on Y . If $Y = X^\delta$ for some degree like function δ on A , then we also make use of notation $\overline{\text{div}}_X^\delta(g)$ for $\overline{\text{div}}_X^Y(g)$.

- If $g \in A$ is such that $\delta_j(g) > 0$ for all $j = 1, \dots, N$, then

$$\delta_g := e_g \cdot \max\left\{\frac{\delta_j}{\delta_j(g)} : 1 \leq j \leq N\right\},$$

where e_g is a suitable integer to ensure that δ_g is integer valued (e.g. one can take $e_g := \prod_{j=1}^N \delta_j(g)$).

Let the filtration corresponding to δ be $\mathcal{F} := \{F_d : d \geq 0\}$. Identify A^δ with $\sum_{d \in \mathbb{Z}} F_d t^d$. Recall that $X^\delta := \text{Proj } A^\delta$ is the union of affine charts of the form $\text{Spec } A_{(gt^d)}^\delta$, where $d > 0$ and $A_{(gt^d)}^\delta$ is the subring of elements of degree zero of the localizations $A_{gt^d}^\delta$. Say U_0, \dots, U_m is an open cover of X^δ with $U_j := \text{Spec } A_{(g_j t^{l_j})}^\delta$ for some $l_j \geq 1$ and $g \in F_{l_j}$ for every j . Moreover, assume that $g_0 = 1$ and $l_0 = 1$, so that $U_0 = \text{Spec } A_{(t)}^\delta = \text{Spec } A$. Let d be a common multiple of l_1, \dots, l_m . Then for each j , $h_j := \frac{t^d}{(g_j)^{d/l_j} t^d}$ is a regular function on U_j and h_j/h_k is a unit on $U_j \cap U_k$. Therefore collection $\{(h_j, U_j)\}_j$ defines an *effective* Cartier divisor $D_{d,\infty}^\delta$ on X^δ , which we call the *d-uple divisor at infinity*. Its associated Weil divisor is

$$[D_{d,\infty}^\delta] := \sum_{j=1}^N \text{ord}_j(D_{d,\infty}^\delta) [V_j],$$

where V_1, \dots, V_N are the irreducible components of X_∞ and ord_j is the shorthand for ord_{V_j} (where ord_{V_j} is as defined in section ??). Support of $[D_{d,\infty}^\delta]$ being X_∞ justifies index ∞ as a subscript of $D_{d,\infty}^\delta$.

Lemma 3.7. *Let X , A , δ and $D_{d,\infty}^\delta$ be as above. Then*

1. $[D_{d,\infty}^\delta] = \sum_{j=1}^N \frac{d}{d_j} [V_j]$ where for every j , integer d_j is the positive generator of the subgroup of \mathbb{Z} generated by $\{\delta_j(f) : f \in A\}$.
2. Let $g \in A$ be such that δ_g is finitely generated. Then the principal divisor of g^d on X^{δ_g} is $[\operatorname{div}_{X^{\delta_g}}(g^d)] = d[\overline{\operatorname{div}}_X^{\delta_g}(g)] - e_g[D_{d,\infty}^{\delta_g}]$.

Proof. 1. Fix integer j , $1 \leq j \leq N$. Local ring $\mathcal{O}_{V_j, X^\delta}$ is a discrete valuation ring and its associated valuation is $\nu_j(\cdot) := -\frac{\delta_j(\cdot)}{d_j}$ (proposition ??). Pick k , $1 \leq k \leq N$, such that $V_j \cap U_k \neq \emptyset$. Recall that $U_k := \operatorname{Spec} A_{(g_k t^{l_k})}^\delta$ and a local equation for $D_{d,\infty}^\delta$ on U_k is $\frac{t^d}{(g_k)^{d/l_k} t^d}$. Let \mathfrak{p}_j be the ideal of A^δ corresponding to V_j . Since $V_j \cap U_k \neq \emptyset$, it follows that $g_k t^{l_k} \notin \mathfrak{p}_j$ and therefore $\delta_j(g_k) = l_k$ according to assertion ?? of lemma ??. Therefore $\operatorname{ord}_j(D_{d,\infty}^\delta) = \nu_j(\frac{t^d}{(g_k)^{d/l_k} t^d}) = \nu_j(1/g_k^{d/l_k}) = -\frac{d}{l_k} \nu_j(g_k) = \frac{d}{l_k} \cdot \frac{l_k}{d_j} = \frac{d}{d_j}$. It follows that $[D_{d,\infty}^\delta] := \sum_{j=1}^N \operatorname{ord}_j(D_{d,\infty}^\delta) [V_j] = \sum_{j=1}^N \frac{d}{d_j} [V_j]$, which completes the proof of assertion 1.

2. Reindexing the δ_j 's if necessary, we may assume that the minimal presentation of δ_g is $\delta_g = \max\{\frac{e_g \cdot \delta_j}{\delta_j(g)} : 1 \leq j \leq M\}$ for some $M \leq N$. For $1 \leq j \leq M$, let V'_j be the irreducible component of $X_\infty^{\delta_g}$ corresponding to δ_j and d'_j be the positive generator of the subgroup of \mathbb{Z} generated by $\{\frac{e_g \cdot \delta_j(h)}{\delta_j(g)} : h \in A\}$. Then, as a straightforward consequence of proposition ?? it follows that $\operatorname{ord}_{V'_j}(g^d) = -\frac{e_g d}{d'_j}$ for every j and therefore $[\operatorname{div}_{X^{\delta_g}}(g^d)] = d[\overline{\operatorname{div}}_X^{\delta_g}(g)] + \sum_{j=1}^M \operatorname{ord}_{V'_j}(g^d) [V'_j] = d[\overline{\operatorname{div}}_X^{\delta_g}(g)] - \sum_{j=1}^M \frac{e_g d}{d'_j} [V'_j]$. On the other hand, applying assertion 1 with δ_g in place of δ yields that $[D_{d,\infty}^{\delta_g}] = \sum_{j=1}^M \frac{d}{d'_j} [V'_j]$. Therefore $[\operatorname{div}_{X^{\delta_g}}(g^d)] = d[\overline{\operatorname{div}}_X^{\delta_g}(g)] - e_g[D_{d,\infty}^{\delta_g}]$, as required. \square

Theorem 3.8. *Assume that $f := (f_1, \dots, f_n) : X \rightarrow \mathbb{K}^n$ and $\delta := \max\{\delta_j : 1 \leq j \leq n\}$ are as in the first paragraph of this section and that $\delta_j(f_i) > 0$, $1 \leq i, j \leq n$. Assume also that for $i = 1, \dots, n$, subdegrees δ_{f_i} are finitely generated. Let $d_{f_i} \geq 1$, $1 \leq i \leq n$, be such that the d_{f_i} -uple embedding of $X^{\delta_{f_i}}$ is a closed immersion of $X^{\delta_{f_i}}$ into a projective space $\mathbb{P}^{L_i}(\mathbb{K})$. Let $L := \prod_{i=1}^n (L_i + 1) - 1$ and \bar{X} be the closure of the image of X in $\mathbb{P}^L(\mathbb{K})$ under the composition of the following maps:*

$$X \hookrightarrow X^{\delta_{f_1}} \times \dots \times X^{\delta_{f_n}} \hookrightarrow \mathbb{P}^{L_1}(\mathbb{K}) \times \dots \times \mathbb{P}^{L_n}(\mathbb{K}) \hookrightarrow \mathbb{P}^L(\mathbb{K}),$$

where the first map is the diagonal embedding and the last map is the Segre embedding. Then for all $a \in \mathbb{K}^n$,

$$|f^{-1}(a)| \leq \frac{e_{f_1} \cdots e_{f_n}}{n^n d_{f_1} \cdots d_{f_n}} \deg(\bar{X}), \quad (\text{C})$$

where $|f^{-1}(a)|$ is the number of the isolated points in $f^{-1}(a)$ each counted with the multiplicity of $f^{-1}(a)$ at the respective point.

Question: Is it true that the completeness of δ implies the finite generation property of every δ_{f_i} ?

Proof of theorem 3.8. Denote by $[y_{i,0} : \cdots : y_{i,L_i}]$ the homogeneous coordinates on $\mathbb{P}^{L_i}(\mathbb{K})$. Without loss of generality we may assume that $X_\infty^{\delta_{f_i}} = X^{\delta_{f_i}} \cap V(y_{i,0})$.

Denote by X^η the closure of the diagonal embedding of variety X into the product $X^{\delta_{f_1}} \times \cdots \times X^{\delta_{f_n}} \subseteq \mathbb{P}^{L_1}(\mathbb{K}) \times \cdots \times \mathbb{P}^{L_n}(\mathbb{K}) =: Y$. Denote by $[y_0 : \cdots : y_L : z_1 : \cdots : z_n]$ the homogeneous coordinates on $\mathbb{P}^{L'}(\mathbb{K})$, where $L' := L + n$. Let us identify $\mathbb{P}^L(\mathbb{K})$ with the subspace $V(z_1, \dots, z_n)$ of $\mathbb{P}^{L'}(\mathbb{K})$ and let $s : \mathbb{P}^{L_1}(\mathbb{K}) \times \cdots \times \mathbb{P}^{L_n}(\mathbb{K}) \rightarrow \mathbb{P}^L(\mathbb{K})$ denote the Segre embedding. Let $s' : X^\eta \rightarrow \mathbb{P}^{L'}(\mathbb{K})$ be the map defined by:

$$s' : X^\eta \ni ([y_{1,0} : \cdots : y_{1,L_1}], \dots, [y_{n,0} : \cdots : y_{n,L_n}]) \mapsto [s(y) : (y_{1,0})^n : \cdots : (y_{n,0})^n] \in \mathbb{P}^{L'}(\mathbb{K}).$$

Fix an i , $1 \leq i \leq n$. Let $D_i := \pi_i^*(D_{d_{f_i}, \infty}^{\delta_{f_i}})$, where $\pi_i : X^\eta \rightarrow X^{\delta_{f_i}}$ is the projection onto the i -th factor of Y . Due to our choice of $y_{i,0}$, Cartier divisor $D_{d_{f_i}, \infty}^{\delta_{f_i}}$ is precisely the restriction of the divisor of $y_{i,0}$ to $X^{\delta_{f_i}}$. It follows that $s'^*(D'_i) = nD_i$, where D'_i is the restriction of the divisor of z_i to $s'(X^\eta)$.

Then $(D_1, \dots, D_n) = \frac{1}{n^n}(s'^*(D'_1), \dots, s'^*(D'_n)) = \frac{1}{n^n}(D'_1, \dots, D'_n)$, since intersection numbers are preserved under the pull backs by proper birational morphisms [2, Example 2.4.3]. Since each D'_i is the divisor of a linear form on $s'(X^\eta)$, it follows that the intersection number $(D'_1, \dots, D'_n) = \deg s'(X^\eta)$. On the other hand, $s|_{X^\eta} = \pi' \circ s'$, where π' is the projection onto the first L coordinates. Since the mapping degree of $\pi'|_{s'(X^\eta)}$ is 1, it follows that $\deg(s(X^\eta)) = \deg s'(X^\eta)$ [11, Proposition 5.5]. Combining three equalities established in this paragraph it follows that $(D_1, \dots, D_n) = \frac{1}{n^n} \deg s(X^\eta)$.

Let $E_i := \pi_i^*(\overline{\text{div}}_X^{\delta_{f_i}}(f_i))$. According to assertion 2 of lemma 3.7 $[\text{div}_{X^{\delta_{f_i}}}(f_i^{d_{f_i}})] = d_{f_i}[\overline{\text{div}}_X^{\delta_{f_i}}(f_i)] - e_{f_i}[D_{d_{f_i}, \infty}^{\delta_{f_i}}]$. Since $\pi_i^*(\text{div}_{X^{\delta_{f_i}}}(f_i)) = \text{div}_{X^\eta}(f_i)$ it follows that

$$(E_1, \dots, E_n) = \frac{e_{f_1} \cdots e_{f_n}}{d_{f_1} \cdots d_{f_n}}(D_1, \dots, D_n) = \frac{e_{f_1} \cdots e_{f_n}}{n^n d_{f_1} \cdots d_{f_n}} \deg s(X^\eta).$$

Fix an i , $1 \leq i \leq n$. Note that $D_{d_{f_i}, \infty}^{\delta_{f_i}}$ are *very ample*, i.e. are the pull backs of the hyperplane sections under the embeddings of $X^{\delta_{f_i}}$ into $\mathbb{P}^{L_i}(\mathbb{K})$. In particular, these divisors are *base point free* [4, Section II.7]. Then the pull backs D_i of $D_{d_{f_i}, \infty}^{\delta_{f_i}}$ are also base point free, and so are $e_{f_i}E_i$ (the latter being linearly equivalent to $d_{f_i}D_i$). Also, since E_i 's are *effective* (defined in section ??), it follows that the intersection number (E_1, \dots, E_n) bounds the sum of the intersection multiplicities of E_i 's at the isolated points of the intersection $\bigcap_{i=1}^n \text{Supp}(E_i)$ [2, Section 12.2]. Of course $X \cap (\bigcap_{i=1}^n \text{Supp}(E_i)) = f^{-1}(0)$. Therefore $|f^{-1}(0)| \leq (E_1, \dots, E_n)$, which completes the proof of the theorem. \square

Future plans:

1. Let $f := (f_1, \dots, f_n) : X \rightarrow \mathbb{K}^n$ be any generically finite map (not necessarily satisfying the hypotheses of theorem 3.8). Replacing f by $\xi \circ f$ for a generic affine transformation ξ of \mathbb{K}^n (and reordering δ_j 's if necessary), one may assume that there is an $M \leq N$ such that

- (a) $\delta_j(f_i) > 0$ for all i and all $j = 1, \dots, M$, and
- (b) $\delta_j(f_i) = 0$ for all i and all $j = M + 1, \dots, N$.

We expect that it should be possible to extend theorem 3.8 in this setting as a consequence of extending the arguments of the proof of theorem 3.8 to the case of $\tilde{X} := \text{Spec } \tilde{A}$, where $\tilde{A} := \{g \in A : \delta_j(g) \leq 0 \text{ for } M + 1 \leq j \leq N\}$, and for the completion $\tilde{X}^{\tilde{\delta}}$ of \tilde{X} determined by $\tilde{\delta} := \max\{\delta_j : 1 \leq j \leq M\}$.

2. Moreover, we hope to prove (by means of an extension of our theorem ?? to the case of subdegrees) that for completion X^δ that preserves map f at ∞ (in the setting of theorem 3.8), inequality in (C) can be replaced by equality.

4 Iterated Semidegrees

In this section we describe particularly simple semidegrees generalizing weighted homogeneous degrees for which we establish a constructive version of affine Bezout-type theorem. Our dream is that a stronger version of Main Existence Theorem ?? would be valid with subdegrees in whose minimal presentations only constructive semidegrees ‘like’ the iterated semidegrees of this section would appear, and we expect that a precise constructive version of an affine Bezout-type theorem for any generically finite map $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$ would follow.

Let A be a domain and δ be a degree like function on A . Pick $f \in A$ and an integer w with $w < \delta(f)$. Let s be an indeterminate over A and δ_e be a ‘natural’ extension of δ to $A[s]$ such that $\delta_e(s) = w$, namely: $\delta_e(\sum a_i s^i) := \max_{a_i \neq 0} (\delta(a_i) + iw)$. Of course δ is a degree like function iff δ_e is a degree like function.

Lemma 4.1. δ is a semidegree iff δ_e is a semidegree.

Proof. The (\Leftarrow) direction is obvious, since $\delta \equiv \delta_e|_A$. For the proof of the ‘only if’ implication, let $G := \sum g_i s^i, H := \sum h_j s^j \in A[s]$ with $d := \delta_e(G), e := \delta_e(H)$. For each $k \geq 0$, let $G_k := \sum_{\delta(g_i) + iw = k} g_i s^i$ and $H_k := \sum_{\delta(h_j) + iw = k} h_j s^j$. It suffices to show that $\delta_e(G_d H_e) = d + e$, and hence we may w.l.o.g. assume $G = G_d$ and $H = H_e$. Let i_0 (resp. j_0) be the largest integer such that $g_{i_0} \neq 0$ (resp. $h_{j_0} \neq 0$). Then $GH = g_{i_0} h_{j_0} s^{i_0 + j_0} + \sum_{m < i_0 + j_0} a_m s^m$. Thus $\delta_e(GH) \geq \delta_e(g_{i_0} h_{j_0} s^{i_0 + j_0}) = \delta(g_{i_0} h_{j_0}) + (i_0 + j_0)w = \delta(g_{i_0}) + i_0 w + \delta(h_{j_0}) + j_0 w = d + e$. Since the inequality $\delta_e(GH) \leq d + e$ is obviously true, it follows that $\delta_e(GH) = d + e$. \square

Remark 4.2. In fact δ is a subdegree iff δ_e is a subdegree, which follows by means of calculations similar to those in the proof of lemma 4.1 and of the characterization of subdegrees as degree like functions η satisfying $\eta(f^k) = k\eta(f)$ for all $f \in A$ and $k \geq 0$ (corollary ??).

Let J denote the ideal generated by $s - f$ in $A[s]$. Identify A with $A[s]/J$ and define $\tilde{\delta}$ to be the degree like function on A induced by δ_e , i.e. $\tilde{\delta}(g) := \min\{\delta_e(G) : G - g \in J\}$. Let \mathfrak{a} be the principal ideal generated by f in A and let \mathfrak{a}^δ be the ideal induced in A^δ by \mathfrak{a} (as defined in lemma 2.1). Denote by $\text{gr } \mathfrak{a}$ the ideal generated by the image of \mathfrak{a}^δ in $\text{gr } A^\delta$, i.e. $\text{gr } \mathfrak{a} := \langle [(g)_{\delta(g)}] : g \in \mathfrak{a} \rangle$, where $[(g)_{\delta(g)}]$ denotes the equivalence class of $(g)_{\delta(g)}$ in $\text{gr } A^\delta$.

Remark 4.3. A straightforward application of definitions shows that $\tilde{\delta}$ is a degree like function provided that δ_e is a degree like function and $\tilde{\delta} \equiv 0$ on \mathbb{K} . Note that $\tilde{\delta}$ is a meaningful degree like function only if $[(f)_{\delta(f)}]$ is not a unit in $\text{gr } A^\delta$. Indeed, if $[(f)_{\delta(f)}]$ is a unit in $\text{gr } A^\delta$, then $[(1)_0] = [(f)_d][(g)_{-d}] \in \text{gr } A^\delta$ for some $g \in A$ with $\delta(g) = -d$. It follows that if $1 - fg \neq 0$ then $\delta(1 - fg) < 0$. Let $G := 1 - fg + gs \in A[s]$. Then $\delta_e(gs) = w - d < 0$ and therefore $\delta_e(G) < 0$. Moreover, $1 \equiv G \pmod{J}$ in $A[s]$. Consequently $\tilde{\delta}(1) \leq \delta_e(G) < 0$. Since, for all $h \in A$ and $n \in \mathbb{Z}_+$, $\tilde{\delta}(h) = \tilde{\delta}(h \cdot (1)^n) \leq \tilde{\delta}(h) + n\tilde{\delta}(1)$, it follows that $\tilde{\delta}(h) = -\infty$ for all $h \in A$. Moreover, $(1)_1$ is a unit in $\mathbb{K}[X]^{\tilde{\delta}}$ (since $((1)_1)^{-1} = (1)_{-1}$) and therefore $\text{gr } \mathbb{K}[X]^{\tilde{\delta}} \cong \mathbb{K}[X]^{\tilde{\delta}} / \langle (1)_1 \rangle$ is the zero ring.

Theorem 4.4 (cf. [7, Example 5] and [8, Theorem 2.1.3]).

1. (a) If δ is non-negative and $w > 0$, then $\tilde{\delta}$ is non-negative.
 (b) $\tilde{\delta}(g) \leq \delta(g)$ for all $g \in A$. If $[(g)_{\delta(g)}] \notin \text{gr } \mathfrak{a}$, then $\tilde{\delta}(g) = \delta(g)$.
 (c) $\tilde{\delta}(f) \leq w < \delta(f)$. If δ is a semidegree and $[(f)_{\delta(f)}]$ is not a unit in $\text{gr } A^\delta$, then $\tilde{\delta}(f) = w$.
2. Assume δ is a semidegree. Then
 - (a) The homomorphism of \mathbb{K} -algebras $A[s]^{\delta_e} \cong A^\delta[s] \rightarrow \mathbb{K}[X]^{\tilde{\delta}}$ induced by the inclusion $A^\delta \hookrightarrow \mathbb{K}[X]^{\tilde{\delta}}$ (available due to $\tilde{\delta} \leq \delta$) and $s \mapsto (f)_w$ is surjective with kernel $\langle (s - f)_{\delta_e(s-f)} \rangle$.
 - (b) $\text{gr } \mathbb{K}[X]^{\tilde{\delta}}$ is isomorphic to $(\text{gr } A^\delta / \text{gr } \mathfrak{a})[z]$, where z is an indeterminate of degree w and the isomorphism maps z to the class of equivalence $[(f)_w]$ of $(f)_w \in \mathbb{K}[X]^{\tilde{\delta}}$.
 - (c) $\tilde{\delta}$ is a semidegree if and only if $\text{gr } \mathfrak{a}$ is a prime ideal of $\text{gr } \mathbb{K}[X]^{\tilde{\delta}}$.

Note that if δ is a semidegree, then $\text{gr } \mathfrak{a}$ is a principal ideal in $\text{gr } A^\delta$ generated by $[(f)_{\delta(f)}]$ due to lemma ??.

Proof. Assertion 1(a) is a straightforward consequence of the definitions. Note that $\tilde{\delta}(g) \leq \delta_e(G)$ for all $g \in A$ and $G \in A[s]$ such that $G \equiv g \pmod{J}$. The first assertion of 1(b) follows by setting $G := g$ in the previous sentence. Similarly, the first assertion of 1(c) follows by setting $g := f$ and $G := s$. As for the second assertion of 1(b), let $g \in A$ and $G \in A[s]$ be such that $G \equiv g \pmod{J}$. Let $a_0, \dots, a_k \in A$ such that $G = g + (s - f)(a_0 + a_1s + \dots + a_k s^k) = (g - fa_0) + \sum_{i=1}^k (a_{i-1} - fa_i)s^i + a_k s^{k+1}$. Note that if $e := \delta(g - fa_0) < d := \delta(g)$, then $\delta(fa_0) = d$ and $(g)_d = (fa_0)_d + ((1)_1)^{d-e}(g - fa_0)_e$, so that $[(g)_d] = [(fa_0)_d] \in \text{gr } \mathfrak{a}$. In other words, if $[(g)_{\delta(g)}] \notin \text{gr } \mathfrak{a}$, then $\delta(g - fa_0) \geq \delta(g)$ and therefore $\delta_e(G) \geq \delta(g)$. It follows that $\tilde{\delta}(g) \geq \delta(g)$, which concludes the proof of 1(b).

Next we prove the second assertion of 1(c). Assume δ is a semidegree and contrary to the conclusion of 1(c) that $\tilde{\delta}(f) < w$. Then it suffices to show that $[(f)_d]$ is a unit in $\text{gr } A^\delta$, where $d := \delta(f)$. Indeed, $\tilde{\delta}(f) < w$ in view of the definition of $\tilde{\delta}$ in terms of δ_e implies that there is an identity

$$f = a_k f^k + a_{k+1} f^{k+1} + \dots + a_l f^l \quad (9)$$

with $a_k, \dots, a_l \in A$ such that for all j , $0 \leq k \leq j \leq l$, $\delta(a_j) + jw < w$. In particular, $\delta(a_0) < w$ if $a_0 \neq 0$ and $\delta(a_1) < 0$ if $a_1 \neq 0$.

If $k > 1$, then dividing both sides of (9) by f it follows that $fg = 1$, where $g := \sum_{j=k}^l a_j f^{j-2}$. Then $\delta(g) = -\delta(f) = -d$ and therefore $[(f)_d] \cdot [(g)_{-d}] = [(1)_0] \in \text{gr } A^\delta$. Since $[(1)_0]$ is the identity in $\text{gr } A^\delta$, it follows that $[(f)_d]$ is a unit in $\text{gr } A^\delta$, which proves assertion 1(c) in the case that $k > 1$.

If $k = 1$, then (9) implies that $1 - a_1 = fg_1$, where $g_1 := a_2 + a_3f + \dots + a_l f^{l-2}$. Since $\delta(a_1) < 0$, it follows that $\delta(1 - a_1) = 0$ and therefore $\delta(g_1) = -d$. Moreover, $[(a_1)_0] = 0 \in \text{gr } A^\delta$. Hence $[(f)_d] \cdot [(g_1)_{-d}] = [(1)_0] \in \text{gr } A^\delta$. Consequently $[(f)_d]$ is a unit in $\text{gr } A^\delta$, as required.

It remains to consider the case of $k = 0$. In this case $a_0 = fg_2$, with element $g_2 := 1 - a_2f - a_3f^2 - \dots - a_l f^{l-1}$. Then $\delta(g_2) = \delta(a_0) - \delta(f) < w - d < 0$ and $g_2 = 1 - fg_1$ with $g_1 := a_2 + a_3f + \dots + a_l f^{l-2}$. Since $\delta(g_2) < 0 = \delta(1)$, it follows that $\delta(fg_1) = \delta(1) = 0$, and therefore $\delta(g_1) = -\delta(f) = -d$. Consequently $[(f)_d] \cdot [(g_1)_{-d}] = [(1)_0] - [(g_2)_0] = [(1)_0] \in \text{gr } A^\delta$, implying $[(f)_d]$ is a unit in $\text{gr } A^\delta$, which completes the proof of 1(c).

Next we prove assertion 2. Assume δ is a semidegree. Due to remark 4.3 it suffices to consider the case when $[(f)_{\delta(f)}]$ is not a unit in $\text{gr } A^\delta$. Then $\delta(f) = w$ according to assertion 1(c). We start with introducing two surjective \mathbb{K} -algebra homomorphisms $\phi : A[s] \rightarrow A$ and $\Phi : A[s]^{\delta_e} \rightarrow \mathbb{K}[X]^{\tilde{\delta}}$ by means of formulae

$$\begin{aligned} \phi\left(\sum_{i=1}^k a_i s^i\right) &:= \sum_{i=1}^k a_i f^i \quad \text{for any } a_1, \dots, a_k \in A, \\ \Phi((H)_d) &:= (\phi(H))_d \quad \text{for all } H \in A[s], \ d \geq \delta_e(H) \in \mathbb{Z}. \end{aligned}$$

Clearly ϕ is surjective and $\ker \phi = J$. It follows that Φ is a surjective homomorphism of graded rings with $\ker \Phi = J^{\delta_e}$, and consequently, $\mathbb{K}[X]^{\tilde{\delta}} = A[s]^{\delta_e} / J^{\delta_e}$. Moreover, δ being a semidegree on A implies that δ_e is also a semidegree on $A[s]$ (lemma 4.1) and therefore $J^{\delta_e} = \langle (s-f)_{\delta_e(s-f)} \rangle$ (lemma ??), which completes the proof of 2(a).

Next we prove assertion 2(b). Ring $\text{gr } \mathbb{K}[X]^{\tilde{\delta}} = \mathbb{K}[X]^{\tilde{\delta}} / \langle (1)_1 \rangle \cong A[s]^{\delta_e} / (J^{\delta_e} + \langle (1)_1 \rangle)$ because $\mathbb{K}[X]^{\tilde{\delta}} = A[s]^{\delta_e} / J^{\delta_e}$. The element z in the assertion of 2(b) (which corresponds to $[(f)_w]$) is precisely the equivalence class $[(s)_w]$ of $(s)_w \in A[s]^{\delta_e}$. Note that the homomorphism defined by $A^\delta[s] \ni \sum (f_i)_{d-iw} s^i \mapsto (\sum f_i s^i)_d \in A[s]^{\delta_e}$ is an isomorphism. Since $\delta_e(s) = w < \delta(f) = \delta_e(f)$, it follows that $(s-f)_{\delta_e(s-f)} = (s)_w(1)_{\delta(f)-w} + (f)_{\delta(f)}$ and therefore $J^{\delta_e} + \langle (1)_1 \rangle = \langle (s-f)_{\delta_e(s-f)}, (1)_1 \rangle = \langle (f)_{\delta(f)}, (1)_1 \rangle$. Hence $\text{gr } \mathbb{K}[X]^{\tilde{\delta}} \cong A^\delta[s] / \langle (f)_{\delta(f)}, (1)_1 \rangle = (A^\delta / \langle (f)_{\delta(f)}, (1)_1 \rangle)[s]$. But $A^\delta / \langle (f)_{\delta(f)}, (1)_1 \rangle \cong (A^\delta / \langle (1)_1 \rangle) / (\langle (f)_{\delta(f)}, (1)_1 \rangle / \langle (1)_1 \rangle)$ and $\langle (f)_{\delta(f)}, (1)_1 \rangle / \langle (1)_1 \rangle$ is precisely the ideal generated by $[(f)_{\delta(f)}]$ in $\text{gr } A^\delta$, which is $\text{gr } \mathfrak{a}$, while $A^\delta / \langle (1)_1 \rangle \cong \text{gr } A^\delta$. It follows that $\text{gr } \mathbb{K}[X]^{\tilde{\delta}} \cong (\text{gr } A^\delta / \text{gr } \mathfrak{a})[s]$, and completes the proof of assertion 2(b).

It remains only to prove assertion 2(c). Due to assertion 2(b), $\text{gr } \mathfrak{a}$ is a prime ideal of $\text{gr } A^\delta$ iff $\text{gr } \mathbb{K}[X]^{\tilde{\delta}}$ is a domain and, of course, iff $\langle (1)_1 \rangle$ is a prime ideal of $\mathbb{K}[X]^{\tilde{\delta}}$. But $\langle (1)_1 \rangle$ is a prime ideal of $\mathbb{K}[X]^{\tilde{\delta}}$ iff $\tilde{\delta}$ is a semidegree (theorem ??), which completes the proof of the theorem. \square

Theorem 4.4 motivates the following:

Definition. Let δ be a semidegree on A . The *leading form* $\mathfrak{L}_\delta(f)$ of an element f of A is the equivalence class $[(f)_{\delta(f)}]$ of $(f)_{\delta(f)}$ in $\text{gr } A^\delta$.

If δ is a semidegree on A and the ideal $\langle \mathfrak{L}_\delta(f) \rangle$ of $\text{gr } A^\delta$ generated by the leading form $\mathfrak{L}_\delta(f)$ of $f \in A$ is prime, then $\tilde{\delta}$ is also a semidegree on A (theorem 4.4). Semidegree $\tilde{\delta}$ differs from δ according to assertion 1. of theorem 4.4. On the other hand, assertion 1(b) of theorem 4.4 shows that $\tilde{\delta}$ agrees with δ off $\text{gr } \mathfrak{a}$. We will say that $\tilde{\delta}$ is formed by the *iteration procedure* starting with semidegree δ by means of $f \in A$.

Example 4.5. Let $A := \mathbb{K}[x_1, x_2]$ and δ be the semidegree on A defined in ??. Recall that $\delta(x_1) = 3$, $\delta(x_2) = 2$ and $\delta(x_1^2 - x_2^3) = 1$. Moreover, \mathbb{K} -algebra A^δ coincides with $\mathbb{K}[(1)_1, (x_1)_3, (x_2)_2, (x_1^2 - x_2^3)_1] = \mathbb{K}[X_1, X_2, Y, Z] / \langle YZ^5 - X_1^2 + X_2^3 \rangle$. We claim that δ is formed by an iteration procedure by means of $f := x_1^2 - x_2^3$ starting with the weighted degree η which assigns weight 3 to x_1 and 2 to x_2 . Indeed, $\text{gr } \mathbb{K}[X]^\eta \cong \mathbb{K}[x_1, x_2]$ via the map that sends $\mathfrak{L}_\eta(h) \in \text{gr } \mathbb{K}[X]^\eta$ to the leading weighted homogeneous component of h . Then, since $f = x_1^2 - x_2^3$ is weighted homogeneous, $\mathfrak{L}_\eta(f) = f$. Since $\mathfrak{L}_\eta(f) = f \in \mathbb{K}[x_1, x_2] = \text{gr } \mathbb{K}[X]^\eta$, it follows that ideal $\langle \mathfrak{L}_\eta(f) \rangle$ is prime. Therefore according to assertion 2. of theorem 4.4,

degree like function $\tilde{\eta}$ formed by the iteration procedure by means of f starting with η is in fact a semidegree. Also $\mathbb{K}[X]^{\tilde{\eta}} = A[s]^{\eta_e} / \langle (s - f)_6 \rangle$, where η_e is the weighted degree on $A[s]$ that extends η and sends s to 1, as defined in the paragraph preceding lemma 4.1. Then with $t := (1)_1$, $A[s]^{\eta_e} / \langle (s - x_1^2 + x_2^3)_6 \rangle = \mathbb{K}[x_1, x_2, s, t] / \langle st^5 - x_1^2 + x_2^3 \rangle \cong A^\delta$. To summarize, semidegree δ of example ?? coincides with the iterated semidegree $\tilde{\eta}$.

Remark 4.6. Assume a semidegree δ on the coordinate ring A of an affine variety X is constructed by means of finitely many iterations starting with a semidegree η . Denote by X^η and X^δ the completions of the d -uple embedding of X into appropriate projective spaces (valid for appropriate $d \in \mathbb{Z}_+$ [10, Lemma in section III.8]). Then we can express the degree of X^δ in terms of the degree of X^η (in a straightforward generalization of theorem 4.8 below). In particular, in the special case of η being a weighted homogeneous degree on $A := \mathbb{K}[x_1, \dots, x_n]$ with weights $0 < d_i := \eta(x_i)$, $1 \leq i \leq n$, $\deg X^\eta = \frac{1}{d_1 \dots d_n}$ (example 3.4) and an explicit formula for $D := \deg X^\delta$, which appears in the affine Bezout-type theorem 3.1, follows:

Theorem 4.7. *Let δ be a complete degree like function on the coordinate ring A of an affine variety X , $f \in A$ and $w \in \mathbb{Z}$ with $0 < w < \delta(f)$. Let δ_e and $\tilde{\delta}$ be degree like functions respectively on $A[s]$ and A defined as above. Finally, let $d \in \mathbb{Z}_+$ be such that both X^δ and $X^{\tilde{\delta}}$ embeds into a usual projective space $\mathbb{P}^l(\mathbb{K})$ via the d -uple embedding, and D (resp. \tilde{D}) be the degree of the image of X^δ (resp. $X^{\tilde{\delta}}$) in $\mathbb{P}^l(\mathbb{K})$. If the ideal $\mathcal{I} := \langle (s - f)_{\delta_e(s-f)} \rangle$ of $A[s]^{\delta_e}$ is prime, then $\tilde{D} = \frac{e}{w} D$.*

Proof. 1. $\mathbb{K}[X]^{\tilde{\delta}} \cong A[s]^{\delta_e} / \mathcal{I}$.

2. The homomorphism defined by $A^\delta[s] \ni \sum (f_i)_{d-iw} s^i \mapsto (\sum f_i s^i)_d \in A[s]^{\delta_e}$ is an isomorphism.

3. Let $e := \delta(f)$ and $t := (1)_1 \in A^\delta$. Then $\mathcal{I} = \langle (f)_e - st^{e-w} \rangle$.

4. Let $(1)_1, (f_1)_{d_1}, \dots, (f_k)_{d_k}$ generate A^δ as a \mathbb{K} -algebra. Then there is a surjection $\Phi : \mathbb{K}[T, Y_1, \dots, Y_k] \twoheadrightarrow A^\delta$, which induces a surjection $\Phi_e : \mathbb{K}[T, Y_1, \dots, Y_k, S] \twoheadrightarrow A^\delta[s] \cong A[s]^{\delta_e}$. Let $\mathcal{J} := \ker \Phi$, and F be a weighted homogeneous polynomial in Y_j 's such that $\tilde{\delta}(F) = e$ and $\Phi(F) = (f)_e$. Then $\ker \Phi_e = \langle \mathcal{J}, F - ST^{e-w} \rangle$.

5. The surjections in the previous steps induces embeddings of the form:

$$\begin{array}{ccc} X^{\tilde{\delta}} & \xrightarrow{\quad} & \mathbf{WP}' := \mathbb{P}^{k+1}(\mathbb{K}; 1, d_1, \dots, d_k, w) \\ \downarrow & & \uparrow \\ Y & & \\ X^{\delta} & \xrightarrow{\quad} & \mathbf{WP} := \mathbb{P}^k(\mathbb{K}; 1, d_1, \dots, d_k) \end{array}$$

Choose d such that the d -uple embedding ψ_d embeds $\mathbb{P}^{k+1}(\mathbb{K}; 1, d_1, \dots, d_k, w)$ into a usual projective space $\mathbb{P}^l(\mathbb{K})$.

6. Let $Y := V(\mathcal{J}) \subseteq \mathbf{WP}'$. Then $\deg \psi_d(Y)$ is the number of intersections of Y with $n+1$ generic hypersurfaces of weighted degree d , which equals $\frac{d}{w} D$.

7. Since $I(X^{\tilde{\delta}}) = I(Y) + \langle F - ST^{e-w} \rangle$, $\deg \psi_d(Y)$ is the number of intersections of Y with n generic hypersurfaces of weighted degree d and $V(F - ST^{e-w})$ which equals $\frac{d}{w} D \times \frac{e}{d} = \frac{e}{w} D$. \square

Corollary 4.8 (see [7, Example 9] and [8, Theorem 3.1.5]). *Let δ_0 be a weighted degree on $A := \mathbb{K}[x_1, \dots, x_n]$. Let $k \geq 1$ and for each $i = 1, \dots, k$, let δ_i be a semidegree on A obtained by an iteration procedure starting with δ_{i-1} by means of a polynomial h_i (with $\langle \mathfrak{L}_{\delta_{i-1}}(h_i) \rangle$ being*

prime in $\text{gr } \mathbb{K}[X]^{\delta_{i-1}}$) by assigning to the polynomial h_i a weight w_i with $0 < w_i < \delta_{i-1}(h_i)$. Then

$$\frac{D}{d^n} = \frac{1}{\delta_0(x_1) \cdots \delta_0(x_n)} \frac{\delta_0(h_1)}{w_1} \cdots \frac{\delta_{k-1}(h_k)}{w_k}, \quad (\text{B})$$

where $D := \deg X^\delta$ and $d \in \mathbb{Z}_+$ are as in theorem 3.1 for $X := \mathbb{K}^n$, $A := \mathbb{K}[x_1, \dots, x_n]$ and $\delta := \delta_k$.

Proof. Let $e_i := \delta_{i-1}(h_i)$ for $1 \leq i \leq k$. According to assertion 2. of theorem 4.4, rings $A^{\delta_i} \cong (A^{\delta_{i-1}}[s_i])^{\delta_{i-1}^e} / \langle (s_i - h_i)_{e_i} \rangle$, where s_i are indeterminates and δ_{i-1}^e extend δ_{i-1} by assigning weights w_i to s_i . It follows by induction on i with $x_0 = (1)_1$ that

$$A^{\delta_i} = \mathbb{K}[x_0, \dots, x_n, s_1, \dots, s_i] / J_i,$$

where $J_i := \langle \tilde{h}_1 - x_0^{e_1 - w_1} s_1, \dots, \tilde{h}_i - x_0^{e_i - w_i} s_i \rangle$, $1 \leq i \leq k$, and $\tilde{h}_j \in \mathbb{K}[\bar{x}, \bar{s}]$ are weighted homogeneous polynomials in $(\bar{x}, \bar{s}) := (x_0, \dots, x_n, s_1, \dots, s_{j-1})$ whose equivalence classes in $\mathbb{K}[X]^{\delta_i}$ are $(h_j)_{e_j}$, $1 \leq j \leq i$.

Let $\tilde{\delta}$ be the weighted degree on $R_k := \mathbb{K}[x_0, \dots, x_n, s_1, \dots, s_k]$ which assigns weight 1 to x_0 , $d_i := \delta_0(x_i)$ to x_i , $1 \leq i \leq n$, and w_j to indeterminates s_j , $1 \leq j \leq k$. Then homomorphism $\pi : R_k / J_k \rightarrow \mathbb{K}[X]^\delta$ of graded \mathbb{K} -algebras is surjective, and, therefore, for each $f \in A$, there is a polynomial \tilde{f} in R_k with $\tilde{\delta}(\tilde{f}) = \delta(f)$ and $\tilde{f} \mapsto (f)_{\delta(f)}$ under homomorphism π . Moreover, homomorphism π induces an embedding of X^{δ_k} into the weighted projective space $\mathbf{WP} := \mathbb{P}^{n+k}(\mathbb{K}; 1, d_1, \dots, d_n, w_1, \dots, w_k)$. Since J_k is generated by exactly k polynomials in R_k , it follows that the image of X^δ in \mathbf{WP} is a *complete intersection*. Identifying X^δ with its image in \mathbf{WP} , it follows that $X^\delta = V(J_k)$ and, therefore, that for any $f_1, \dots, f_n \in \mathbb{K}[x_1, \dots, x_n]$

$$\bigcap_{i=1}^n \{x \in \mathbb{K}^n : f_i(x) = 0\} \subseteq X^\delta \cap V(\tilde{f}_1) \cap \cdots \cap V(\tilde{f}_n) = V(\tilde{h}_1 - x_0^{e_1 - w_1} s_1) \cap \cdots \cap V(\tilde{h}_k - x_0^{e_k - w_k} s_k) \cap V(\tilde{f}_1) \cap \cdots \cap V(\tilde{f}_n). \quad (10)$$

Arguing via an embedding of $\mathbf{WP} \hookrightarrow \mathbb{P}^N(\mathbb{K})$ it suffices to choose as f_j 's the pull backs of generic linear polynomials on $\mathbb{P}^N(\mathbb{K})$ and then the intersection on the right hand side of the equality in (10) would consist of isolated points in $X := \mathbb{K}^n \hookrightarrow X^\delta \hookrightarrow \mathbb{P}^N(\mathbb{K})$ (of multiplicities one and of the total number being the degree of X^δ in $\mathbb{P}^N(\mathbb{K})$ according to the commonly used geometric definition of degree of a projective variety [3, Definition 18.1]). Consequently due to weighted homogeneous Bezout theorem (example 3.4) on \mathbb{K}^{n+k} , the sum of the intersection multiplicities of Cartier divisors corresponding to $\tilde{h}_i - x_0^{e_i - w_i} s_i$, $1 \leq i \leq k$, and \tilde{f}_j , $1 \leq j \leq n$, at the points in the left hand side of the equality in (10) is $\frac{\delta(f_1) \cdots \delta(f_n) e_1 \cdots e_k}{d_1 \cdots d_n w_1 \cdots w_k}$, and by the Bezout theorem for semidegrees, the sum of the multiplicities of the fiber $f^{-1}(0)$ at the points in the left hand side of the inclusion in (10) is $\frac{D}{d^n} \delta(f_1) \cdots \delta(f_n)$. Formula (B) follows by comparing these two expressions, which completes the proof. \square

Example 4.9. Let $f_k := (x_1 + (x_1^2 - x_2^2)^2, (x_1^2 - x_2^2)^k) : \mathbb{K}^2 \rightarrow \mathbb{K}^2$. We estimate the size of fibers of $f_k =: (f_{k1}, f_{k2})$ in three different ways. The first one is by means of the weighted homogeneous Bezout formula (8). It is straightforward to see that the smallest upper bound given by (8) for f_k is achieved for $d_1 = 3p$ and $d_2 = 2p$ for some $p \geq 1$, in which case the bound is $\frac{12p \cdot 6kp}{3p \cdot 2p} = 12k$. The second approach we take is via Bernstein's theorem (see section

??). Let $a := (a_1, a_2) \in \mathbb{K}^2$ and let \mathcal{P} and \mathcal{Q}_k be the Newton polygons of $x_1 + (x_1^2 - x_2^3)^2 - a_1$ and, respectively, of $(x_1^2 - x_2^3)^k - a_2$. The BKK bound for $|f_k^{-1}(a)|$ for non-zero a_1, a_2 is then $2\mathcal{M}(\mathcal{P}, \mathcal{Q}) = \text{Vol}(\mathcal{P} + \mathcal{Q}) - \text{Vol}(\mathcal{P}) - \text{Vol}(\mathcal{Q}) = \frac{1}{2}(2k+4)(3k+6) - 12 - 3k^2 = 12k$. (A similar argument implies that the BKK bound for a_1 or a_2 being zero is as well $12k$.)

Let δ be the iterated semidegree on $\mathbb{K}[x_1, x_2]$ from example 4.5, so that $\delta(x_1) = 3$, $\delta(x_2) = 2$ and $\delta(x_1^2 - x_2^3) = 1$. Then $D/d^n = \frac{6}{3 \cdot 2 \cdot 1} = 1$ (theorem 4.8). The estimate of $|f_k^{-1}(a)|$ given by (A) with this δ is then $\delta(x_1 + (x_1^2 - x_2^3)^2)\delta((x_1^2 - x_2^3)^k) = 3k$. Moreover, ideals generated by $(f_{k1})_{\delta(f_{k1})}$, $(f_{k2})_{\delta(f_{k2})}$ and $(1)_1$ in A^δ are primary to the irrelevant ideal of A^δ and, therefore, completion ψ_δ preserves f_k at ∞ over all points in \mathbb{K}^2 . In other words our bound with iterated semidegree δ is exact!

5 Dimension 2 Revisited

In this section we continue the exploration of the relation of the number of solutions of a system of polynomials with subdegrees that preserve the system when $\dim X = 2$. At first we settle in this case question ?? of section ?? in the affirmative.

Lemma 5.1. *If δ is a degree like function and η is a semidegree such that $\delta \geq \eta$, then the ideal $\mathfrak{p}_{\delta, \eta}$ of A^δ generated by $\{(f)_d : d > \eta(f)\}$ is a prime ideal of A^δ . Moreover, if δ is a subdegree and η is not an associated semidegree of δ , then $V(\mathfrak{p}_{\delta, \eta})$ has codimension at least 2 in X^δ .*

Proof. Let $L := \{(f)_d : d > \eta(f)\}$. Since $\eta \leq \delta$, it follows that if $(f)_d \in L$ and $(g)_e \in A^\delta$, then $(f)_d(g)_e = (fg)_{d+e} \in L$. This implies that L is precisely the set of homogeneous elements of $\mathfrak{p}_{\delta, \eta}$. Therefore, if $(f_1)_{d_1}, (f_2)_{d_2} \in A^\delta \setminus \mathfrak{p}_{\delta, \eta}$, then $\eta(f_i) = d_i$ for each i , $1 \leq i \leq 2$. It follows that $\eta(f_1 f_2) = d_1 + d_2$ and hence $(f_1 f_2)_{d_1 d_2} \in A^\delta \setminus \mathfrak{p}_{\delta, \eta}$. Therefore $\mathfrak{p}_{\delta, \eta}$ is prime.

Now assume δ is a subdegree with minimal presentation $\delta = \max\{\delta_j : 1 \leq j \leq N\}$. Let \mathfrak{p}_j be the prime ideal of A^δ corresponding to δ_j . Since $(1)_1 \in \mathfrak{p}_{\delta, \eta}$, it follows that $\mathfrak{p}_{\delta, \eta} \supseteq \mathfrak{p}_j$ for some j .

Claim 5.1.1. *For each j , $1 \leq j \leq N$, $\mathfrak{p}_{\delta, \eta} = \mathfrak{p}_j$ iff $\eta = \delta_j$.*

Proof. The ‘if’ direction follows directly from assertion ?? of lemma ??. We now show the ‘only if’ direction. Assume $\mathfrak{p}_{\delta, \eta} = \mathfrak{p}_j$. There exists $(f)_d \in A^\delta \setminus \mathfrak{p}_j$ such that $\delta_j(f) > \delta_i(f)$ for all $i \neq j$. Let $g \in A$. Then there exists $k \in \mathbb{N}$ such that $\delta_j(f^k g) = \delta(f^k g)$. Since $\mathfrak{p}_{\delta, \eta} = \mathfrak{p}_j$, it follows that $\eta(f) = \delta(f)$ and $\eta(f^k g) = \delta(f^k g)$. Therefore $\eta(g) = \eta(f^k g) - \eta(f^k) = \delta(f^k g) - \delta(f^k) = \delta_j(f^k g) - \delta_j(f^k) = \delta_j(g)$. \square

By the above claim it follows that $\mathfrak{p}_{\delta, \eta} \supsetneq \mathfrak{p}_j$ for all j , which completes the proof of the lemma. \square

Do you need X to be normal to ensure that $[\overline{\text{div}}_X^{\delta_g}(g)]$ is an effective CARTIER divisor?? I do not think so - look at the normalization \overline{X}^{δ_g} of X . If $[\overline{\text{div}}_X^{\delta_g}(g)]$ is not effective as a Cartier divisor on \overline{X}^{δ_g} (i.e. at some point its local equation is not regular), then it necessarily has at least one pole Y and it has to be contained in $\overline{X}^{\delta_g} \setminus X$. Then the image of Y is one of the components V_j of infinity in X^{δ_g} . It follows that $\mathcal{O}_{Y, \overline{X}^{\delta_g}} \supseteq \mathcal{O}_{V_j, X^{\delta_g}}$ and hence by maximality of discrete valuation rings $\mathcal{O}_{Y, \overline{X}^{\delta_g}} = \mathcal{O}_{V_j, X^{\delta_g}}$. It follows that order of vanishing of the local equation of $[\overline{\text{div}}_X^{\delta_g}(g)]$ along Y is 0 - a contradiction.

Proposition 5.2. *Let $\delta = \max\{\delta_j : 1 \leq j \leq N\}$ and $g \in A$ such that $\delta_j(g) > 0$ for all j and both δ and δ_g are finitely generated. Pick positive integers $r \leq N$ and m_1, \dots, m_r such that δ' is a finitely generated subdegree with minimal presentation $\delta' := \max\{m_j \delta_j : 1 \leq j \leq r\}$. Let $\phi : X^{\delta'} \rightarrow X^{\delta_g}$ be the birational map induced by identification of X and let $S \subseteq X^{\delta'}$ be the set of points of indeterminacy of ϕ . Then for all j , $1 \leq j \leq r$, $\phi(V'_j \setminus S) \subseteq V(\mathfrak{p}_{\delta_g, \tilde{\delta}_j})$ where V'_j is the component of the hypersurface at infinity of $X^{\delta'}$ corresponding to $m_j \delta_j$, $\tilde{\delta}_j := \frac{e_g}{\delta_j(g)} \delta_j$ and $\mathfrak{p}_{\delta_g, \tilde{\delta}_j}$ is as defined in lemma 5.1.*

Proof. Assume contrary to the proposition that there exists j , $1 \leq j \leq r$, such that $\phi(V'_j \setminus S) \not\subseteq V(\mathfrak{p}_{\delta_g, \tilde{\delta}_j})$. Pick $x \in \phi(V'_j \setminus S) \setminus V(\mathfrak{p}_{\delta_g, \tilde{\delta}_j})$. Let $(f)_d \in \mathfrak{p}_{\delta_g, \tilde{\delta}_j}$ such that $x \notin V((f)_d)$.

Note that $x \in X^{\delta_g} \setminus X = V((1)_1)$ and therefore for a suitable positive integer k , we may assume that the local equation of the kd -uple divisor $D_{d_{kd}, \infty}^{\delta_g}$ at infinity of X^{δ_g} on a neighborhood U of x is $\frac{1}{f^k}$. According to assertion 2 of lemma 3.7, $kd[\overline{\text{div}}_X^{\delta_g}(g)] = [\text{div}_{X^{\delta_g}}(g^{kd})] + e_g[D_{d_{kd}, \infty}^{\delta_g}]$ is a Cartier divisor on X^{δ_g} . By construction the local equation of $kd[\overline{\text{div}}_X^{\delta_g}(g)]$ on U is $\frac{g^{kd}}{f^{ke_g}}$. Note that

$$\begin{aligned} \delta_j\left(\frac{g^{kd}}{f^{ke_g}}\right) &= kd\delta_j(g) - ke_g\delta_j(f) \\ &= k\delta_j(g)\left(d - \frac{e_g\delta_j(f)}{\delta_j(g)}\right) \\ &= k\delta_j(g)(d - \tilde{\delta}_j(f)) \end{aligned}$$

By assumption on δ , $\delta_j(g) > 0$. Moreover $d > \tilde{\delta}_j(f)$, since $(f)_d \in \mathfrak{p}_{\delta_g, \tilde{\delta}_j}$. It follows that $\delta_j(\frac{g^{kd}}{f^{ke_g}}) > 0$ and hence $\frac{g^{kd}}{f^{ke_g}}$ has a pole at V'_j , so that the pullback of $kd[\overline{\text{div}}_X^{\delta_g}(g)]$ to $X^{\delta'} \setminus S$ is not effective. But this is impossible, since $kd[\overline{\text{div}}_X^{\delta_g}(g)]$ is an effective Cartier divisor. This contradiction proves the proposition. \square

Proposition 5.3. *Let X , δ and g be as in proposition 5.2. Moreover assume $\dim X = 2$. Let λ_j^i , $1 \leq i \leq k$, $1 \leq j \leq N$, be positive integers and X^η be the closure of the diagonal embedding of X into $X^{\delta_g} \times X^{\delta^1} \times \dots \times X^{\delta^k}$, where $\delta^i := \max\{\lambda_j^i \delta_j : 1 \leq j \leq N\}$ for $1 \leq i \leq k$. Then $\pi^*(\overline{\text{div}}_X^{\delta_g}(g)) = \overline{\text{div}}_X^\eta(g)$, where π is the projection in the first coordinate.*

Proof. Let $\overline{V(g)}^{\delta_g} \cap X^{\delta_g} \setminus X = \{x_1, \dots, x_r\}$, where $\overline{V(g)}^{\delta_g}$ is the closure in X^{δ_g} of $V(g) \subseteq X$. Now, $[\pi^*(\overline{\text{div}}_X^{\delta_g}(g))] = [\overline{\text{div}}_X^\eta(g)] + E$, where $\text{Supp } E \subseteq X^\eta \setminus X$. Assume $E \neq \emptyset$. Then E is of the form $\sum_{j=1}^s m_j[V_j]$ such that for each j , $1 \leq j \leq s$, $\pi(V_j) = x_{i_j}$ for some i_j , $1 \leq i_j \leq r$. WLOG we may assume $i_1 = 1$. Since $\dim(V_1) = 1$, there exists j , $1 \leq j \leq k$, such that $\dim(\pi_j(V_1))$ is also 1, where $\pi_j : X^\eta \rightarrow X^{\delta^j}$ is the natural projection map. WLOG assume $j = 1$ and let $\phi : X^{\delta^1} \rightarrow X^{\delta_g}$ be the birational map induced by the identification of X in both spaces and $S \subseteq X^{\delta^1}$ be the set of points of indeterminacy of ϕ . Since $\pi \equiv \phi \circ \pi_1$ on $X^\eta \setminus \pi_1^{-1}(S)$, it follows that $\phi(\pi_1(V_1) \setminus S) = \{x_1\}$.

WLOG we may order $\delta_1, \dots, \delta_N$ in a way that δ^1 has minimal presentation $\delta^1 = \max\{\lambda_j^1 \delta_j : 1 \leq j \leq M_1\}$ for some $M_1 \leq N$ and $\pi_1(V_1)$ is the component of the hypersurface at infinity of X^{δ^1} corresponding to $\lambda_1^1 \delta_1$. Henceforth we write V'_1 for $\pi_1(V_1)$. Let $\tilde{\delta}_j := \frac{e_g}{\delta_j(g)} \delta_j$ for $1 \leq j \leq N$. Then $\delta_g = \max\{\tilde{\delta}_1, \dots, \tilde{\delta}_N\}$.

Claim. $\tilde{\delta}_1$ is not an associated semidegree of δ_g .

Proof. Assume $\tilde{\delta}_1$ is an associated semidegree of δ_g . Then according to claim 5.1.1 $\mathfrak{p}_{\delta_g, \tilde{\delta}_1} = \tilde{\mathfrak{p}}_1$, where $\tilde{\mathfrak{p}}_1$ is the prime ideal of A^{δ_g} corresponding to $\tilde{\delta}_1$. Proposition 5.2 then implies that $\phi(V'_1 \setminus S) \subseteq \tilde{V}_1 := V(\tilde{\mathfrak{p}}_1)$. Since $\tilde{\delta}_1$ and $\lambda_1^1 \delta_1$ induces the same discrete valuation ν on $\mathbb{K}(X)$, it follows due to proposition ?? that $\mathcal{O}_{V'_1, X^{\delta_1}} = \mathcal{O}_{\tilde{V}_1, X^{\delta_g}}$, both of these rings being same as the valuation ring of ν . It follows that $\overline{\phi(V'_1 \setminus S)} = \tilde{V}_1$, which is absurd, since $\phi(V'_1 \setminus S) = \{x_1\}$. This contradiction proves the claim. \square

Since $\tilde{\delta}_1$ is not an associated semidegree of δ_g , lemma 5.1 implies that $V(\mathfrak{p}_{\delta_g, \tilde{\delta}_1})$ has codimension 2 in X^{δ_g} . Since $V(\mathfrak{p}_{\delta_g, \tilde{\delta}_1})$ is also irreducible, it follows that $V(\mathfrak{p}_{\delta_g, \tilde{\delta}_1})$ is a single point. Let x be the sole element of $V(\mathfrak{p}_{\delta_g, \tilde{\delta}_1})$.

Claim. $x \notin \overline{V(g)}^{\delta_g}$.

Proof. Since $\tilde{\delta}_j(g) = e_g$ for all j , $1 \leq j \leq N$, it follows that $\overline{V(g)}^{\delta_g} = V((g)_{e_g})$ (remark ??). Moreover, $\tilde{\delta}_1(g) = e_g = \delta_g(g)$, so that $(g)_{e_g} \notin \mathfrak{p}_{\delta_g, \tilde{\delta}_1}$. Therefore $x \notin V((g)_{e_g}) = \overline{V(g)}^{\delta_g}$, as required. \square

According to proposition 5.2 $\{x_1\} = \phi(V'_1 \setminus S) \subseteq V(\mathfrak{p}_{\delta_g, \delta_1}) = \{x\}$. But this contradicts the above claim. It follows that $E = \emptyset$ and hence $[\pi^*(\overline{\text{div}}_X^{\delta_g}(g))] = [\overline{\text{div}}_X^{\eta}(g)]$, as required. \square

Corollary 5.4. Assume $\dim X = 2$ and that $f := (f_1, f_2) : X \rightarrow \mathbb{K}^2$, $\delta := \max\{\delta_j : 1 \leq j \leq N\}$ and $d_{f_1}, d_{f_2} \in \mathbb{N}$ are as in theorem 3.8. If X^δ preserves $\{f_1, f_2\}$ at ∞ then for almost all $a \in X$, (C) holds with an equality.

Proof. Note that for each $a \in X$, replacing f by $f - a$ does not affect the assumptions on f . Therefore it suffices to show that if X^δ preserves $\{f_1, f_2\}$ at ∞ over 0 then (C) holds with an equality for $a = 0$.

Let $X^{\eta'}$ (resp. X^η) be the closure of the diagonal embedding of X into $X^\delta \times X^{\delta_{f_1}} \times X^{\delta_{f_2}}$ (resp. $X^{\delta_{f_1}} \times X^{\delta_{f_2}}$). Then there is a system of maps as follows:

$$\begin{array}{ccccc} & & X^{\eta'} & & \\ & \swarrow \pi' & & \searrow \pi'_0 & \\ & X^\eta & & & \\ \swarrow \pi_1 & & & \searrow \pi_2 & \\ X^{\delta_{f_1}} & & & & X^{\delta_{f_2}} \end{array}$$

such that each map is the identity on X . Fix an i , $1 \leq i \leq 2$. Let $D_i := \pi_i^*(D_{d_{f_i}, \infty}^{\delta_{f_i}})$ and $D'_i := \pi'^*(D_i) = \pi'_i^*(D_{d_{f_i}, \infty}^{\delta_{f_i}})$, where $\pi'_i := \pi_i \circ \pi'$. Due to lemma 3.7, on $X^{\eta'}$, $[\text{div}_{X^{\eta'}}(f_i^{d_{f_i}})] = d_{f_i}[\pi'_i^*(\overline{\text{div}}_X^{\delta_{f_i}}(f_i))] - e_{f_i}[D'_i]$. Since $\pi'_i^*(\overline{\text{div}}_X^{\delta_{f_i}}(f_i)) = \overline{\text{div}}_X^{\eta'}(f_i)$ according to proposition 5.3, it follows that $[\text{div}_{X^{\eta'}}(f_i^{d_{f_i}})] = d_{f_i}[\overline{\text{div}}_X^{\eta'}(f_i)] - e_{f_i}[D'_i]$ and therefore

$$(D'_1, D'_2) = \frac{d_{f_1} d_{f_2}}{e_{f_1} e_{f_2}} (\overline{\text{div}}_X^{\eta'}(f_1), \overline{\text{div}}_X^{\eta'}(f_2)) .$$

By assumption X^δ preserves $\{f_1, f_2\}$ at ∞ . It follows that $X^{\eta'}$ also preserves $\{f_1, f_2\}$ at ∞ . Therefore the points at the intersection of supports of $\overline{\text{div}}_X^{\eta'}(f_1)$ and $\overline{\text{div}}_X^{\eta'}(f_2)$ are precisely the points of $f^{-1}(0)$ a intersections of $V(f_1)$ and $V(f_2)$. This implies that

$$|f^{-1}(0)| = (\overline{\text{div}}_X^{\eta'}(f_1), \overline{\text{div}}_X^{\eta'}(f_2)) = \frac{e_{f_1} e_{f_2}}{d_{f_1} d_{f_2}}(D'_1, D'_2) . \quad (11)$$

Moreover intersection numbers are preserved under the pull backs by proper birational morphisms [2, Example 2.4.3] and therefore $(D'_1, D'_2) = (\pi'^*(D_1), \pi'^*(D_2)) = (D_1, D_2)$. Recall that $(D_1, D_2) = \deg(s(X^\eta))$ according to (??), where s is the Segre embedding of X^η into the product of ambient spaces of $X^{\delta_{f_1}}$ and $X^{\delta_{f_2}}$. Combining the latter equality with (11), we obtain the desired equality. \square

Lemma 5.5. *Let η_1, \dots, η_k be complete degree like functions on A . Then there is a complete degree like function η and proper maps $\phi_i : X^\eta \rightarrow X^{\eta_i}$ for $i = 1, \dots, k$ such that the following diagram commutes for each i :*

$$\begin{array}{ccc} X^\eta & \xrightarrow{\phi_i} & X^{\eta_i} \\ \uparrow \psi_\eta & & \uparrow \psi_{\eta_i} \\ X & \xrightarrow{\mathbf{1}} & X \end{array}$$

Proof. Clear: let X^η be the closure of the diagonal embedding of X into $X^{\eta_1} \times \dots \times X^{\eta_k}$. \square

Assume for all $\lambda := (\lambda_1, \dots, \lambda_N)$ with $\lambda_j \geq 1$, $\delta^{(\lambda)} := \max\{\lambda_j \delta_j : 1 \leq j \leq N\}$ is finitely generated. Let $\{\lambda^i : i \geq 1\}$ be an enumeration of \mathbb{N}^2 . For each i , let $g_i := (f_1)^{\lambda_1^i d_{f_1}} (f_2)^{\lambda_2^i d_{f_2}}$. As above, define quasidegree δ_{g_i} on A , choose a suitable $d_{g_i} \in \mathbb{N}$ and let $D_{\delta_{g_i}, d_{g_i}}$ be the divisor at ∞ on $X^{\delta_{g_i}}$. By lemma 5.5, there exist completions X_1, X_2, \dots of X and a system of maps as follows:

$$\begin{array}{ccccccc} & X^{\delta_{g_i}} & X^{\delta_{g_{i-1}}} & & X^{\delta_{g_2}} & X^{\delta_{g_1}} & \\ & \uparrow & \uparrow & & \uparrow & \uparrow & \\ \cdots & \rightarrow X_i & \rightarrow X_{i-1} & \rightarrow \cdots & \rightarrow X_2 & \rightarrow X_1 & \rightarrow X^\delta \\ & & & & & \swarrow & \searrow \\ & & & & & X^{\delta_{f_1}} & \cdots & X^{\delta_{f_n}} \end{array}$$

Fix $K \geq 1$. For each $j \leq K$ and each $k \geq K$, we can pull back on X_k the Cartier divisor $D_{\delta_{g_j}, d_{g_j}}$ and we can also pull back each $D_{\delta_{f_i}, d_{f_i}}$ for $i = 1, 2$. If D_1, D_2 are Cartier divisors on X_k which are linear combinations of those in the preceeding sentence, the intersection product (D_1, D_2) is independent of k [?, Example 2.4.3]. In particular, $(D_{\delta_{g_j}, d_{g_j}})^2$ and $(D_{\delta_{f_{i_1}}, d_{f_{i_1}}}, D_{\delta_{f_{i_2}}, d_{f_{i_2}}})$ are well defined for each $j \leq 1$ and $1 \leq i_1, i_2 \leq n$. Now fix any j . Let V_{g_j} (resp. V_{f_i} for $1 \leq i \leq 2$) be the (principal) Cartier divisor on X generated by g_j (rep. f_i). Then $V_{g_j} = \sum_{i=1}^2 \lambda_i^j d_{f_i} V_{f_i}$. Applied to X_j , lemma 3.7 implies that $((f_i)^{d_{f_i}}) = d_{f_i} \bar{V}_{f_i} - D_{\delta_{f_i}, d_{f_i}}$ for each i , so that $(g_j) = \sum_{i=1}^2 \lambda_i^j d_{f_i} (f_i) = \sum_{i=1}^2 \lambda_i^j (d_{f_i} \bar{V}_{f_i} - D_{\delta_{f_i}, d_{f_i}}) = \bar{V}_{g_j} - \sum_{i=1}^2 \lambda_i^j D_{\delta_{f_i}, d_{f_i}}$. By lemma 3.7 again, it follows that $D_{\delta_{g_j}, d_{g_j}} = d_{g_j} \sum_{i=1}^2 \lambda_i^j D_{\delta_{f_i}, d_{f_i}}$. Therefore, the function $\mathcal{M} : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by:

$$\mathcal{M}(\lambda^j) := \frac{(D_{\delta_{g_j}, d_{g_j}})^2}{(d_{g_j})^2}$$

depends polynomially on its arguments, since

$$\mathcal{M}(\lambda_1, \lambda_2) = \left(\sum_{i=1}^2 \lambda_i D_{\delta_{f_i}, d_{f_i}} \right)^2 = \lambda_1^2 D_{\delta_{f_1}, d_{f_1}}^2 + 2\lambda_1 \lambda_2 (D_{\delta_{f_1}, d_{f_1}}, D_{\delta_{f_2}, d_{f_2}}) + \lambda_2^2 D_{\delta_{f_2}, d_{f_2}}^2. \quad (12)$$

We say that ν *separates* $\delta_1, \dots, \delta_N$ if for all $\lambda_1, \dots, \lambda_N \in \mathbb{N}$, $\bigcap_{i=1}^N \nu(F_k^{\lambda_i \delta_i}) = \nu(\bigcap_{i=1}^N F_k^{\lambda_i \delta_i})$ for all sufficiently large k .

Example 5.6. Let \prec be any total ordering on \mathbb{Z}^n such that it is compatible with addition. Define $\nu : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{Z}^n$ by: $\nu(\sum_{\alpha} a_{\alpha} x^{\alpha}) := \min_{\prec} \{\alpha : a_{\alpha} \neq 0\}$. Then ν separates all the weighted degrees in x_1, \dots, x_n coordinates.

Example 5.7. Let $w_1 := x - y^2$ and $w_2 := x + y^2$. Then $\mathbb{C}[x, y] = \mathbb{C}[w_1, y] = \mathbb{C}[w_2, y]$. Let δ_i be the degree in (w_i, y) coordinates, $1 \leq i \leq 2$, and ν be any valuation on $\mathbb{C}[x, y]$ induced by an ordering of monomials in (x, y) . Then \prec does *not* separate δ_1 and δ_2 . Indeed, $F_1^{\delta_1} = \mathbb{C}\langle 1, y, x - y^2 \rangle$ and $F_1^{\delta_2} = \mathbb{C}\langle 1, y, x + y^2 \rangle$. Consequently, $F_1^{\delta_1} \cap F_1^{\delta_2} = \mathbb{C}\langle 1, y \rangle$ and $\nu(F_1^{\delta_1} \cap F_1^{\delta_2}) = \{(0, 0), (0, 1)\}$. On the other hand, $\nu(F_1^{\delta_1}) \cap \nu(F_1^{\delta_2}) = \{(0, 0), (0, 1), (1, 0)\}$.

Now let X, δ, f be as in Theorem 3.8 and let ν be a valuation on $A := \mathbb{K}[X]$ that *separates* $\delta_1, \dots, \delta_N$. Let C_j be the smallest closed cone in \mathbb{R}^3 containing

$$G_j := \left\{ \left(\frac{1}{d_{g_j}} \delta_{g_j}(h), \nu(h) \right) \in \mathbb{Z}_+^3 : h \in A \right\}.$$

For each $\lambda \in \mathbb{N}^n$, let C^{λ} be the smallest cone in \mathbb{R}^3 containing

$$G^{\lambda} := \left\{ \left(\frac{1}{d_{g_j}} \delta_{g_j}(h), \nu(h) \right) \in \mathbb{Z}_+^3 : h \in A \right\}.$$

By lemma 3.7 and proposition 3.5, $(D_{\delta_{g_j}, d_{g_j}})^2 = 2 \text{Vol}(\Delta_j)$, where Δ_j is the convex hull of the cross-section of C_j at the first coordinate value 1. Since ν separates $\delta_1, \dots, \delta_N$, it follows that $C_j = C_{j,1} \cap \dots \cap C_{j,N}$, where for each k , $C_{j,k}$ be the smallest closed cone in \mathbb{R}^3 containing

$$G_{j,k} := \left\{ \left(\frac{1}{d_{g_j} \delta_k(g_j)} \delta_k(h), \nu(h) \right) \in \mathbb{Z}_+^3 : h \in A \right\}.$$

It follows that

$$\mathcal{M}(\lambda_1, \lambda_2) = \text{Vol} \left(\bigcap_{j=1}^N (\lambda_1 d_{1j} + \lambda_2 d_{2j}) \Delta^{(j)} \right)$$

For appropriate d_{ij} and $\Delta^{(j)}$'s.

Now, [6, Theorem 6.4] implies that each $\Delta^{(j)}$ has *linear* edges. Since \mathcal{M} is a homogeneous polynomial of degree 2 in λ_i 's, this forces that the sums and intersection commute in the preceding expression for \mathcal{M} , i.e.

$$\mathcal{M}(\lambda_1, \lambda_2) = \text{Vol} \left(\lambda_1 \bigcap_{j=1}^N d_{1j} \Delta^{(j)} + \lambda_2 \bigcap_{j=1}^N d_{2j} \Delta^{(j)} \right).$$

It follows from comparing the preceding expression with (12), that $(D_{\delta_{f_1}, d_{f_1}}, D_{\delta_{f_2}, d_{f_2}})$ is precisely the mixed volume of $\bigcap_{j=1}^N d_{1j} \Delta^{(j)}$ and $\bigcap_{j=2}^N d_{2j} \Delta^{(j)}$, as required.

Remark. The theorem has to be stated explicitly!

References

- [1] James Damon. A global weighted version of Bezout’s theorem. In *The Arnoldfest (Toronto, ON, 1997)*, volume 24 of *Fields Inst. Commun.*, pages 115–129. Amer. Math. Soc., Providence, RI, 1999.
- [2] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [3] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. A first course.
- [4] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [5] Kiumars Kaveh and A. G. Khovanskii. Convex bodies and algebraic equations on affine varieties. arXiv:[math.AG], 2008.
- [6] R. Lazarsfeld and M. Mustata. Convex bodies associated to linear series. arXiv:[math.AG], 2008.
- [7] Pinaki Mondal. An affine Bezout type theorem and projective completions of affine varieties. *C. R. Math. Acad. Sci. Soc. R. Can.*, 30(4):1–12, 2008.
- [8] Pinaki Mondal. On projective completions of affine varieties determined by ‘degree-like’ functions. arXiv:0901.4823v1 [math.AG], 2009.
- [9] Pinaki Mondal. Projective completions of affine varieties determined via ‘degree-like’ functions. arXiv:1012.0835 [Math.AG], 2010.
- [10] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
- [11] David Mumford. *Algebraic geometry. I*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Complex projective varieties, Reprint of the 1976 edition.